Theories with NIP. List 9.

We work in a monster model $\mathfrak{C}$ of a complete theory T.

Problem 1. Prove that T has NTP_2 if and only if $\kappa_{inp}(T) < \infty$.

Problem 2. Prove that if $M \models T$ and $p \in S(M)$ is definable, then p has a unique global heir.

Problem 3. Prove that any model M (of any theory T) is an extension base.

Problem 4. Let $A \subseteq \mathfrak{C}$ and $p \in S(A)$. Prove that p has a global heir which does not fork over A if and only if the type $\pi(x) := p(x) \cup \pi_1(x) \cup \pi_2(x)$ is consistent, where:

- $\pi_1(x) := \{\psi(x, b) \in L(\mathfrak{C}) : \psi(x, b') \in p$ for all b' from $A\}$,
- $\pi_2(x) := \{\varphi(x, c) \in L(\mathfrak{C}) : \neg\varphi(x, c)$ divides over $A\}$.

Show that $\pi_1(x)$ is closed under finite conjunctions.

Problem 5. Let $A \subseteq B \subseteq \mathfrak{C}$ and $a \in \mathfrak{C}$. Let $p = \mathrm{tp}(a/B)$. Prove that:

(i) p is strictly non-forking over A if and only if the type $\pi(x) := p(x) \cup \pi_1(x) \cup \pi_2(x)$ is consistent, where:

- $\pi_1(x) := \{\psi(x, b) \in L(\mathfrak{C}) : \neg\psi(a, y)$ forks over $A\}$,
- $\pi_2(x) := \{\varphi(x, c) \in L(\mathfrak{C}) : \neg\varphi(x, c)$ divides over $A\}$;

(ii) the definition of p being strictly non-forking over A does not depend on the choice of $\mathfrak{C}$;

(iii) $a \mathop{\smile\!\!\!\!\!\mid}^{st}_A B$ if and only if $a \mathop{\smile\!\!\!\!\!\mid}^{st}_A B_0$ for all finite $B_0 \subseteq B$.

Show that $\pi_1(x)$ is closed under finite conjunctions.

Problem 6. Let $M \models DLO$ and $p \in S_1(M)$ be non-algebraic. Let $a < b < c$ realize p. Prove that:

(i) $\mathrm{tp}(ac/Mb)$ does not fork over M and that it is not strictly non-forking over M;

(ii) $\mathrm{tp}(ac/M)$ has exactly two extensions over Ab which are strictly non-forking over M.

Problem 7. Assume NIP. Use Theorem 1 from page 49 of the notes to show that every complete type over a model M is strictly non-forking over M.
Comment. In other words, one could say that in NIP theories every model is an extension base for strict non-forking.