

ON STABLE QUOTIENTS

KRZYSZTOF KRUPIŃSKI AND ADRIÁN PORTILLO

ABSTRACT. We solve two problems from [HP18] concerning maximal stable quotients of groups \wedge -definable in NIP theories. The first result says that if G is a \wedge -definable group in a distal theory, then $G^{st} = G^{00}$ (where G^{st} is the smallest \wedge -definable subgroup with G/G^{st} stable, and G^{00} is the smallest \wedge -definable subgroup of bounded index). In order to get it, we prove that distality is preserved under passing from T to the hyperimaginary expansion T^{heq} . The second result is an example of a group G definable in a non-distal, NIP theory for which $G = G^{00}$ but G^{st} is not an intersection of definable groups. Our example is a saturated extension of $(\mathbb{R}, +, [0, 1])$. Moreover, we make some observations on the question whether there is such an example which is a group of finite exponent. We also take the opportunity and give several characterizations of stability of hyperdefinable sets, involving continuous logic.

1. INTRODUCTION

The core of model-theory is stability theory, developed in the 70's and 80's of the previous century. In the past three decades, one of the main goals of model theory has become finding extensions of stability theory to various unstable contexts, covering many mathematically interesting examples. One either tries to impose some general global assumptions on the theory in question (such as NIP or simplicity) or some local ones (e.g. work with a stable definable set or generically stable type), and prove some structural results. It is also natural and ubiquitous in model theory to look at a “global-local” situation, namely quotients by \wedge -definable (i.e. type-definable) equivalence relations and assume (or prove) their good properties (e.g. boundedness) to get some further conclusions. Recall that a *hyperimaginary* is a class of a \wedge -definable equivalence relation, and a *hyperdefinable set* is a quotient of a \wedge -definable set by such a relation. While bounded quotients have played an important role in model theory and its applications (e.g. to approximate subgroups) for many years, stable quotients have not been studied thoroughly. They appeared in [HP18]. However, it is folklore that hyperimaginaries can be treated as imaginaries in continuous logic via a definable pseudometric (see [BY10a; CH21] and [Han20, Chapter 3] in the language of continuous logic and [BY05] in the language of CATs), so in a sense stability of hyperdefinable sets is equivalent to stability (of imaginary sorts) in continuous logic developed in [BYU10; BY10b]. This is an additional motivation to consider stable quotients. So in Section 2 we take the opportunity and give several characterizations of stability of hyperdefinable sets in

2020 *Mathematics Subject Classification.* 03C45.

Key words and phrases. Stable quotient, hyperimaginary, distal theory, model-theoretic components.

Both authors are supported by the Narodowe Centrum Nauki grant no. 2016/22/E/ST1/00450. The first author is also supported by the Narodowe Centrum Nauki grant no. 2018/31/B/ST1/00357.

various terms, involving continuous logic, including generically stable types which seem to be not considered (or even defined) so far in this context.

In the main parts of the paper, however, we will study stability of hyperdefinable sets (mostly groups) without referring to continuous logic, just using the definition from [HP18] which we recall below, or a characterization via bounds on the number of types observed in Section 2.

Let T be a complete theory, $\mathfrak{C} \models T$ a monster (i.e. κ -saturated and strongly κ -homogeneous for a strong limit cardinal $> |T|$) model in which we are working, and $A \subset \mathfrak{C}$ a small set of parameters (i.e. $|A| < \kappa$); a cardinal γ is *bounded* if $\gamma < \kappa$.

Definition 1.1. *A hyperdefinable (over A) set X/E is stable if for every A -indiscernible sequence $(a_i, b_i)_{i < \omega}$ with $a_i \in X/E$ for all (equivalently, some) $i < \omega$, we have*

$$\text{tp}(a_i, b_j/A) = \text{tp}(a_j, b_i/A)$$

for all (some) $i \neq j < \omega$.

Let G be a \emptyset - \wedge -definable group. There is always a smallest A - \wedge -definable subgroup of G of bounded index, which is denoted by G_A^{00} . Under NIP, the group G_A^{00} does not depend on the choice of A (see [She08]) and is denoted by G^{00} . So G^{00} is the smallest \wedge -definable (over parameters) subgroup of G of bounded index, and it is in fact \emptyset - \wedge -definable and normal. Staying in the NIP context, G^0 is defined as the intersection of all relatively definable subgroups of bounded index, and it turns out to be \emptyset - \wedge -definable and normal. Regarding stable quotients, since stability of hyperdefinable sets is closed under taking products and type-definable subsets (see [HP18, Remark 1.4]), it is clear that there always exists a smallest A - \wedge -definable subgroup G_A^{st} such that the quotient G/G_A^{st} is stable. The main result of [HP18] says that under NIP, G_A^{st} does not depend on A , and so it is the smallest \wedge -definable (over parameters) subgroup with stable quotient G/G^{st} , and it is in fact \emptyset - \wedge -definable and normal. Under NIP, there is also a \emptyset - \wedge -definable subgroup $G^{st,0}$ which is defined as the intersection of all relatively definable (with parameters) subgroups H of G such that G/H is stable. It is interesting to study those canonical “components” as well as quotients by them. To give a non-stable example, consider a monster model K of ACVF, and $G := (V, +)$, where V is the valuation ring of K . Then $G^{st} = G^{st,0}$ is precisely the additive group of the maximal ideal of V , and G/G^{st} is the additive group of the residue field.

In [HP18], the authors suggested that it should be true that for groups definable in o-minimal theories, and, more generally, in distal theories (see Definition 3.2), $G^{st} = G^{00}$. This agrees with the intuition that distality should be thought of as something at the opposite pole from stability. As an illustration, consider the unit circle in the monster model of RCF: then $G^{st} = G^{00}$ is the group of infinitesimals and $G^0 = G$. In Section 3, we prove this conjecture in the following more general form (see Corollary 3.5).

Proposition 1.2. *If T is distal, then every stable hyperdefinable set is bounded.*

This is deduced from the following result (see Theorem 3.4).

Proposition 1.3. *If T is distal, then T^{hea} is distal (by which we mean that all dense indiscernible sequences of hyperimaginaries are distal).*

We prove the above proposition by elaborating on some arguments from [Sim13].

By Hrushovski's theorem (i.e. [Pil96, Ch. 1, Lemma 6.18]), we know that a \wedge -definable group in a stable theory is an intersection of definable groups. However, although G/G^{st} is stable, it may happen that G^{st} is not an intersection of relatively definable subgroups of G , e.g. in the above example with the unite circle, $G^{st} = G^{00}$ is not an intersection of definable groups. In [HP18], the authors stated as a problem to find an example of a definable group G where $G^{00} = G$ but $G^{st} \neq G^{st,0}$ (i.e. G^{st} is not an intersection of definable groups). In Section 4, we give such an example: it is the monster model of $\text{Th}((\mathbb{R}, +, [0, 1])$. It is not clear to us, however, how to find an example of a torsion (equivalently, finite exponent) group G with those properties, or just satisfying $G^{00} \neq G^{st} \neq G^{st,0}$. In Section 5, we make some observations on this problem, describing what should be constructed in order to find such an example. Dropping the requirement that G is a torsion group, we give a large class of examples where $G^0 \neq G^{00} \neq G^{st} \neq G^{st,0}$; this does not include an example of finite exponent, as G being of finite exponent implies that $G^0 = G^{00}$ by general topological reasons (i.e. compact torsion groups are profinite).

2. CHARACTERIZATIONS OF STABILITY OF HYPERDEFINABLE SETS

Let T be a complete, first order theory, and $\mathfrak{C} \models T$ a monster model. Let E be a \emptyset - \wedge -definable equivalence relation on a \emptyset - \wedge -definable subset X of \mathfrak{C}^λ (or a product of sorts), where $\lambda < \kappa$ (from the definition of \mathfrak{C}). Recall that E is said to be *bounded* if $|X/E| < \kappa$.

In this section, we give some characterizations of stability of the hyperdefinable set X/E , analogous to classical characterizations of stability of a first order theory. This involves continuous logic (CL). Assuming NIP, we also give a characterization using generically stable types, which we introduce in the context of X/E (which in fact could be also done in a general CL context).

Since finite models are trivially stable, we will assume that T has infinite models.

It is folklore that E yields a pseudometric (or a set of pseudometrics) on X (see [BY10a; CH21] and [Han20, Chapter 3] in the language of continuous logic and [BY05] in the language of CATs), which in turn leads to a presentation of X/E as a type-definable set of imaginaries in the sense of continuous logic. Note that in this translation hyperdefinable sets do not translate to continuous logic definable sets. However, for our purposes, it is more convenient to look at the connection with continuous logic in a different way.

We will focus on the first order theory T and treat it as a continuous logic theory, as the aim of this paper is talk about X/E rather than develop continuous logic in general. We will be using some results from [BYU10] but also the formalism from [HKP22, Subsection 3.1] and [HKP21, Section 3]. In particular, by a *CL-formula over A* we mean a continuous function $\varphi: S_n(A) \rightarrow \mathbb{R}$. If φ is such a CL-formula, then for any $\bar{b} \in M^n$ (where $M \models T$) by $\varphi(\bar{b})$ we mean $\varphi(\text{tp}(\bar{b}/A))$; note that the range of every CL-formula is compact. So a CL-formula can be thought of as a function from \mathfrak{C}^n to \mathbb{R} which factors through $S_n(A)$ via a continuous map $S_n(A) \rightarrow \mathbb{R}$. What are called *definable predicates*, in finitely many variables and without parameters, in [BYU10] are precisely CL-formulas over \emptyset , but where the range is contained in $[0, 1]$. In any case, a CL-formula can be added as a new CL-predicate and then it becomes a legitimate formula in the sense of continuous logic. It is not so if we allow the domain of a CL-formula to be an infinite Cartesian power of \mathfrak{C} (which is necessary to deal with X/E in the case when λ is infinite), but

still the results from [BYU10] which we will be using are valid for such generalized continuous logic formulas.

Let M be a model, and $\varphi(x, y)$ a CL-formula over M . Let $a \in \mathfrak{C}^{|x|}$. Then $\text{tp}_\varphi(a/M)$ is the function taking $b \in M^{|y|}$ (or $\varphi(x, b)$) to $\varphi(a, b)$, and is called a *complete $\varphi(x, y)$ -type over M* . The space of all complete φ -types over M is denoted by $S_\varphi(M)$ (it is naturally a quotient of $S(M)$, and the topology on $S_\varphi(M)$ is the quotient topology). The type $\text{tp}_\varphi(a/M)$ is *definable* if it is the restriction to $M^{|y|}$ of a CL-formula $\psi(y)$ over M , i.e. $\varphi(a, b) = \psi(b)$ for $b \in M^{|y|}$.

From now on, let $\mathcal{F}_{X/E}$ be the family of all functions $f : X \times \mathfrak{C}^m \rightarrow \mathbb{R}$ which factor through $X/E \times \mathfrak{C}^m$ and can be extended to a CL-formula $\mathfrak{C}^\lambda \times \mathfrak{C}^m \rightarrow \mathbb{R}$ over \emptyset , where m ranges over ω . (Note that, by Tietze extension theorem, a function $f : X \times \mathfrak{C}^m \rightarrow \mathbb{R}$ extends to a CL-formula over \emptyset iff it factors through the type space $S_{X \times \mathfrak{C}^m}(\emptyset)$ via a continuous function $S_{X \times \mathfrak{C}^m}(\emptyset) \rightarrow \mathbb{R}$.) For $f \in \mathcal{F}_{X/E}$, a *complete f -type over M* is the function taking $f(x, b)$ (for $b \in M^{|y|}$) to $f(a, b)$ for some fixed $a \in X$, and is denoted by $\text{tp}_f(a/M)$. We get the space $S_f(M)$ of all complete f -types over M . A *complete $\mathcal{F}_{X/E}$ -type over M* is the union $\bigcup_{f \in \mathcal{F}_{X/E}} \text{tp}_f(a/M)$ for some $a \in X$, and $S_{\mathcal{F}_{X/E}}(M)$ is the space of all complete $\mathcal{F}_{X/E}$ -types over M . The definition of $\text{tp}_f(a/M)$ being definable is the same as in the previous paragraph; a type in $S_{\mathcal{F}_{X/E}}(M)$ is *definable* if its restriction to any $f \in \mathcal{F}_{X/E}$ is definable.

Let $A \subset \mathfrak{C}$ (be small). Recall that the complete types over A of elements of X/E can be defined as the $\text{Aut}(\mathfrak{C}/A)$ -orbits on X/E , or the preimages of these orbits under the quotient map, or the partial types defining these preimages, or the classes of the equivalence relation on $S_X(A)$ given in the proof of Remark 2.5. The space of all such types is denoted by $S_{X/E}(A)$.

Proposition 2.1. *For any $a_1 = a'_1/E$, $a_2 = a'_2/E$ in X/E and $b_1, b_2 \in \mathfrak{C}^m$*

$$\text{tp}(a_1, b_1) \neq \text{tp}(a_2, b_2) \iff (\exists f \in \mathcal{F}_{X/E})(f(a'_1, b_1) \neq f(a'_2, b_2))$$

Proof. Let us define an equivalence relation E' on $X \times \mathfrak{C}^m$ by

$$(x_1, y_1)E'(x_2, y_2) \iff (x_1/E, y_1) \equiv (x_2/E, y_2).$$

Note that E' is a \emptyset - \wedge -definable, bounded equivalence relation.

(\Leftarrow) Assume $r_1 := f(a'_1, b_1) \neq f(a'_2, b_2) =: r_2$ for some $f \in \mathcal{F}_{X/E}$. Since the sets $f^{-1}(r_1)$ and $f^{-1}(r_2)$ are \emptyset - \wedge -definable and they are unions of $(E \times \{=\})$ -classes, they are unions of E' -classes. But they are also disjoint. Hence, (a'_1, b_1) is not E' -related to (a'_2, b_2) , i.e. $\text{tp}(a_1, b_1) \neq \text{tp}(a_2, b_2)$.

(\Rightarrow) Since E' is \emptyset - \wedge -definable and bounded, $(X \times \mathfrak{C}^m)/E'$ is a compact (Hausdorff) topological space (with the *logic topology*, in which closed sets are those whose preimages by the quotient map are \wedge -definable). Since we assume that $\text{tp}(a_1, b_1) \neq \text{tp}(a_2, b_2)$, we have $[(a'_1, b_1)]_{E'} \neq [(a'_2, b_2)]_{E'}$ in $(X \times \mathfrak{C}^m)/E'$. The space $(X \times \mathfrak{C}^m)/E'$ is $T_{3+\frac{1}{2}}$, so the above two distinct points can be separated by a continuous function

$$h : (X \times \mathfrak{C}^m)/E' \rightarrow \mathbb{R}$$

such that $h([(a'_1, b_1)]_{E'}) = 0$ and $h([(a'_2, b_2)]_{E'}) = 1$. Let $\pi_{E'} : X \times \mathfrak{C}^m \rightarrow (X \times \mathfrak{C}^m)/E'$ be the quotient map. We conclude that the function

$$f := h \circ \pi_{E'} : X \times \mathfrak{C}^m \rightarrow \mathbb{R}$$

satisfies the required conditions. \square

We say that $f \in \mathcal{F}_{X/E}$ is *stable* if for all $\varepsilon > 0$ there do not exist a_i, b_i for $i < \omega$ with $a_i \in X$ for each i , such that for all $i < j$, $|f(a_i, b_j) - f(a_j, b_i)| \geq \varepsilon$ (see [BYU10, Definition 7.1] and [HKP22, Definition 3.8]). By Ramsey's theorem and compactness, f is stable iff whenever $(a_i, b_i)_{i < \omega}$ is indiscernible (with $a_i \in X$), then $f(a_i, b_j) = f(a_j, b_i)$ for all (some) $i < j$.

Corollary 2.2. *X/E is stable as a hyperdefinable set if and only if every $f \in \mathcal{F}_{X/E}$ is stable.*

Proof. (\Rightarrow) Suppose that there is an unstable $f \in \mathcal{F}_{X/E}$. Then there is an indiscernible sequence $(a_i, b_i)_{i < \omega}$ with $a_i \in X$ such that

$$f(a_i, b_j) \neq f(a_j, b_i)$$

for all $i < j$. Hence, by Proposition 2.1, $\text{tp}(a_i/E, b_j) \neq \text{tp}(a_j/E, b_i)$ for all $i < j$. Since the sequence $(a_i/E, b_i)_{i < \omega}$ is indiscernible, we conclude that X/E is not stable.

(\Leftarrow) Suppose that X/E is not stable. Then, there is an indiscernible sequence $(a_i/E, b_i)_{i < \omega}$ with $a_i \in X$ such that

$$\text{tp}(a_i/E, b_j) \neq \text{tp}(a_j/E, b_i).$$

for all $i < j$. By Ramsey's theorem and compactness, we can assume that the sequence $(a_i, b_i)_{i < \omega}$ is indiscernible.

By Proposition 2.1, we conclude that there is $f \in \mathcal{F}_{X/E}$ such that $f(a_i, b_j) \neq f(a_j, b_i)$ for all $i < j$. Hence, f is not stable. \square

The next result follows from [BYU10, Proposition 7.7] and its proof. However, one should be a bit careful here. In the case when λ is finite and $X = \mathfrak{C}^\lambda$, one just applies [BYU10, Proposition 7.7], but in general one should say that the proof of [BYU10, Proposition 7.7] goes through working with $f \in \mathcal{F}_{X/E}$ in place of a legitimate continuous logic formula φ . Also, since we are working in the first order theory T treated as a continuous logic theory, models are discrete spaces and the density characters of models are just cardinalities. The density character of $S_f(M)$ (denoted by $\|S_f(M)\|$) is computed with respect to a certain metric on $S_f(M)$ defined after Definition 6.1 in [BYU10].

Fact 2.3 ([BYU10], Proposition 7.7). *Let $f \in \mathcal{F}_{X/E}$. The following conditions are equivalent:*

- (1) f is stable.
- (2) For every $M \models T$, every $p \in S_f(M)$ is definable.
- (3) For every $M \models T$, $\|S_f(M)\| \leq |M|$.
- (4) For every $M \models T$, $|S_f(M)| \leq |M|^{\aleph_0}$.
- (5) There is $\mu \geq |T|$ such that when $M \models T$ and $|M| \leq \mu$, then $\|S_f(M)\| \leq \mu$.
- (6) For every $\mu = \mu^{\aleph_0} \geq |T|$, when $M \models T$ and $|M| \leq \mu$, then $|S_f(M)| \leq \mu$.

Corollary 2.4. *The following conditions are equivalent:*

- (1) $\forall f \in \mathcal{F}_{X/E}$ f is stable.
- (2) $\forall M \models T \forall f \in \mathcal{F}_{X/E} \forall p \in S_f(M)$ p is definable.
- (3) $\exists \mu \geq |T|$ s.t. $\forall M \models T$ if $M \models T$ and $|M| \leq \mu$, then $|S_{\mathcal{F}_{X/E}}(M)| \leq \mu$.
- (4) $\forall \mu = \mu^{|T|+\lambda} \geq |T| \forall M \models T$ if $M \models T$ and $|M| \leq \mu$, then $|S_{\mathcal{F}_{X/E}}(M)| \leq \mu$.

Proof. This follows easily from Fact 2.3. Only (1) \Rightarrow (4) is a bit more delicate, which we will explain. So assume (1). Then we have (6) from Fact 2.3.

Fix $m < \omega$. By Stone-Weierstrass theorem, the first order formulas restricted to $X \times \mathfrak{C}^m$ generate a dense subalgebra \mathcal{A}_m of cardinality at most $|T| + \lambda$ of the Banach algebra \mathcal{B}_m of all functions $f : X \times \mathfrak{C}^m \rightarrow \mathbb{R}$ which extend to a CL-formula from $\mathfrak{C}^\lambda \times \mathfrak{C}^m$ to \mathbb{R} . As the family $\mathcal{F}_{X/E}^m$ of those functions from \mathcal{B}_m which factor through $X/E \times \mathfrak{C}^m$ is a subspace of \mathcal{B}_m , it also has a dense subset \mathcal{D}_m of cardinality at most $|T| + \lambda$. Since clearly $\mathcal{F}_{X/E} = \bigcup_{m < \omega} \mathcal{F}_{X/E}^m$, we get that the complete $\mathcal{F}_{X/E}$ -type over M of an element $a \in \mathfrak{C}^\lambda$ is determined by $\bigcup_{m < \omega} \bigcup_{f \in \mathcal{D}_m} \text{tp}_f(a/M)$. Using this and (6) from Fact 2.3, one easily gets (4) in Corollary 2.4. \square

Remark 2.5. *For any model M of T there is a natural bijection*

$$S_{X/E}(M) \rightarrow S_{\mathcal{F}_{X/E}}(M).$$

Proof. $S_{\mathcal{F}_{X/E}}(M)$ can be seen as $S_X(M) / \sim_{\mathcal{F}_{X/E}}$, where for every $p, q \in S_X(M)$ and some (equivalently, any) $a'_1 \models p$ and $a'_2 \models q$:

$$p \sim_{\mathcal{F}_{X/E}} q \iff (\forall f(x, y) \in \mathcal{F}_{X/E})(\forall b \in M^{|y|})(f(a'_1, b) = f(a'_2, b)).$$

On the other hand, $S_{X/E}(M) = S_X(M) / \sim_E$, where for every $p, q \in S_X(M)$ and some (equivalently, any) $a'_1 \models p$ and $a'_2 \models q$:

$$p \sim_E q \iff a'_1/E \equiv_M a'_2/E \iff (\forall m < \omega)(\forall b \in M^m)((a'_1/E, b) \equiv (a'_2/E, b)).$$

By Proposition 2.1, $p \sim_{\mathcal{F}_{X/E}} q$ iff $p \sim_E q$. Hence, the conclusion follows. \square

From the previous results, we get some characterizations of stability of X/E .

Corollary 2.6. *The following conditions are equivalent:*

- (1) X/E is stable.
- (2) $\forall f \in \mathcal{F}_{X/E}$ (f is stable).
- (3) $\forall M \models T \forall f \in \mathcal{F}_{X/E} \forall p \in S_f(M)$ (p is definable).
- (4) $\exists \mu \geq |T| \forall M \models T (|M| \leq \mu \implies |S_{\mathcal{F}_{X/E}}(M)| \leq \mu)$.
- (5) $\exists \mu \geq |T| \forall M \models T (|M| \leq \mu \implies |S_{X/E}(M)| \leq \mu)$.
- (6) $\forall \mu = \mu^{|T|+\lambda} \geq |T| \forall M \models T (|M| \leq \mu \implies |S_{X/E}(M)| \leq \mu)$.

Proof. The equivalence between (1), (2), (3), and (4) follows from Corollaries 2.2 and 2.4. The equivalence of (4) and (5) follows from Remark 2.5. The equivalence of (2) and (6) follows from Corollary 2.4 and Remark 2.5. \square

As an application of the characterization from Corollary 2.6(6), we give a quick proof of Remark 2.5(iii) from [HP18] that G^{st} does not have proper hyperdefinable, stable quotients (which was left to the reader in [HP18]). Namely, suppose $H < G^{st}$ is a proper A - \wedge -definable subgroup for some A ; add all elements of A as new constants. We need to show that G^{st}/H is unstable. By minimality of G^{st} , G/H is unstable. So, by Corollary 2.6(6), there is $\mu = \mu^{|T|+\lambda} \geq |T|$, a model M of T of cardinality μ , and a sequence $(g_i)_{i < \mu^+}$ in G such that $\text{tp}(g_i H/M) \neq \text{tp}(g_j H/M)$ for all $i \neq j$. Since G/G^{st} is stable, by Corollary 2.6(6), there is a subset I of μ^+ of cardinality μ^+ such that $g_i G^{st} \equiv_M g_j G^{st}$ for all $i, j \in I$. Fix $i_0 \in I$ and put $I_0 := I \setminus \{i_0\}$. Mapping all $g_i, i \in I_0$, by automorphisms over M , we can assume that they are all in the coset $g_{i_0} G^{st}$. Then $g'_i := g_{i_0}^{-1} g_i \in G^{st}$ for all $i \in I_0$. Moreover,

take any $N \succ M$ containing g_{i_0} and with $|N| = \mu$. Then $\text{tp}(g'_i H/N) \neq \text{tp}(g'_j H/N)$ for every distinct $i, j \in I_0$. Hence, by Corollary 2.6, G^{st}/H is unstable.

Next, we recall the definition of NIP for a hyperdefinable set, given in [HP18, Remark 2.3], and we introduce the notion of generic stability for hyperimaginary types.

Definition 2.7. *A hyperdefinable set X/E has NIP if there do not exist an indiscernible sequence $(b_i)_{i < \omega}$ and $d \in X/E$ such that $((d, b_{2i}, b_{2i+1}))_{i < \omega}$ is indiscernible and $\text{tp}(d, b_0) \neq \text{tp}(d, b_1)$. (Note that the b_i can be anywhere, not necessarily in X/E .)*

Let $p \in S_{X/E}(\mathfrak{C})$ be invariant over A . A *Morley sequence* in p over A is a sequence $(a_i)_i$ of elements of X/E such that $a_i \models p|_{Aa_{<i}}$. As in the home sort, by a standard argument, one can check that Morley sequences (of a given length) in p over A are A -indiscernible and have the same type over A .

Definition 2.8. *An A -invariant type $p \in S_{X/E}(\mathfrak{C})$ is generically stable if every Morley sequence $(a_i/E)_{i < \omega + \omega}$ in p over A satisfies $(\forall \varepsilon > 0) (\forall r \in \mathbb{R}) (\forall s \leq r - \varepsilon) (\forall f(x, y) \in \mathcal{F}_{X/E}) (\forall b \in \mathfrak{C}^{|y|})$*

$$\{i < \omega + \omega : f(a_i, b) \leq s\} \text{ is finite}$$

or

$$\{i < \omega + \omega : f(a_i, b) \geq r\} \text{ is finite.}$$

Generic stability of p does not depend on the choice of A over which p is invariant. Using compactness theorem, one can show the following characterization.

Proposition 2.9. *An A -invariant type $p \in S_{X/E}(\mathfrak{C})$ is generically stable if and only if for every $\varepsilon > 0$ and $f(x, y) \in \mathcal{F}_{X/E}$ there exists $N(f, \varepsilon) \in \mathbb{N}$ for which there is no Morley sequence $(a_i/E)_{i < \omega}$ in p over A , subsequences R, S each of which of length at least $N(f, \varepsilon)$, and $b \in \mathfrak{C}^{|y|}$ such that $|f(a_i, b) - f(a_j, b)| \geq \varepsilon$ for all $a_i/E \in R$ and $a_j/E \in S$.*

Our next goal is to extend Corollary 2.6 to:

Theorem 2.10. *Assume X/E has NIP. The following conditions are equivalent:*

- (1) X/E is stable.
- (2) $\forall M \models T \forall f \in \mathcal{F}_{X/E} \forall p \in S_f(M)$ (p is definable).
- (3) $\exists \lambda \geq |T| \forall M \models T (|M| \leq \lambda \implies |S_{X/E}(M)| \leq \lambda)$.
- (4) Any indiscernible sequence of elements of X/E is totally indiscernible.
- (5) Any global invariant (over some A) type $p \in S_{X/E}(\mathfrak{C})$ is generically stable.

From the proof of this theorem, it will be clear that (1), (2), (3), and (5) are equivalent and imply (4) without the NIP assumption; NIP is used to prove the implication from (4) to (1).

In order to prove Theorem 2.10, we will first prove some results about hyperdefinable sets with NIP and about generically stable types.

From now on, EM will stand for Ehrenfeucht-Mostowski. By the *EM-type* of a sequence $I = (a_i/E)_{i \in \mathcal{I}}$ (symbolically, $\text{EM}(I)$) we mean the set of all formulas $\varphi(x_1, \dots, x_n)$ such that for every $i_1 < \dots < i_n \in \mathcal{I}$, $\varphi(a'_{i_1}, \dots, a'_{i_n})$ holds for all $a'_{i_1} \in [a_{i_1}]_E, \dots, a'_{i_n} \in [a_{i_n}]_E$, where n ranges over ω . We say that an indiscernible sequence J satisfies the EM-type of I if $\text{EM}(I) \subseteq \text{EM}(J)$. By Ramsey's theorem and

compactness, for every sequence I there is an indiscernible sequence J satisfying $\text{EM}(I)$; we can even require that J has an indiscernible sequence of representatives.

In the next three lemmas, X/E and Y/F are arbitrary \emptyset -hyperdefinable sets.

Lemma 2.11. *The following conditions are equivalent:*

- (1) *There exists an indiscernible sequence $(b_i)_{i < \omega}$ in Y/F and $d \in X/E$ such that $((d, b_{2i}, b_{2i+1}))_{i < \omega}$ is indiscernible and $\text{tp}(d, b_0) \neq \text{tp}(d, b_1)$.*
- (2) *There exists an indiscernible sequence $(b_i)_{i < \omega}$ in Y/F and $d \in X/E$ such that*

$$\text{tp}(d, b_i) = \text{tp}(d, b_0) \iff i \text{ even}$$

$$\text{tp}(d, b_i) = \text{tp}(d, b_1) \iff i \text{ odd.}$$

Proof. (1) \Rightarrow (2) is clear.

(2) \Rightarrow (1) Assume (2). Then $\text{tp}(d, b_0) \neq \text{tp}(d, b_1)$. As $(b_i)_{i < \omega}$ is indiscernible, so is $(b_{2i}, b_{2i+1})_{i < \omega}$. Choose an indiscernible sequence $(\tilde{d}, b_{2i}, b_{2i+1})_{i < \omega}$ satisfying the EM-type of $(d, b_{2i}, b_{2i+1})_{i < \omega}$. Then, $\tilde{d} \in X/E$ and $(b_i)_{i < \omega}$ witness (1). \square

Lemma 2.12. *Let $p, q \in S_{X/E \times Y/F}(\emptyset)$ be distinct types. Then the following conditions are equivalent:*

- (1) *There exists an indiscernible sequence $(b_i)_{i < \omega}$ in Y/F and an element $d \in X/E$ with:*

$$\text{tp}(d, b_i) = p \iff i \text{ even}$$

$$\text{tp}(d, b_i) = q \iff i \text{ odd.}$$

- (2) *There is a sequence $(b_i)_{i < \omega}$ in Y/F (not necessarily indiscernible) which is shattered by (p, q) in the sense that for every $I \subseteq \omega$ there is $d_I \in X/E$ with*

$$\text{tp}(d_I, b_i) = p \iff i \in I$$

$$\text{tp}(d_I, b_i) = q \iff i \notin I.$$

Proof. (1) \Rightarrow (2) Let $(b_i)_{i < \omega}$ in Y/F and $d \in X/E$ witness (1). Let $I \subseteq \omega$. We can find an increasing one to one map $\tau : \omega \rightarrow \omega$ such that for all $i \in \omega$, $\tau(i)$ is even iff $i \in I$. By indiscernibility, the map sending b_i to $b_{\tau(i)}$ for all $i \in \omega$ can be extended to an automorphism σ . The element $d_I := \sigma^{-1}(d)$ satisfies the conditions in (2).

(2) \Rightarrow (1) Let $(b_i)_{i < \omega}$ witness (2). We can find an indiscernible sequence $(c_i)_{i < \omega}$ in Y/F satisfying the EM-type of $(b_i)_{i < \omega}$. It follows that for any two disjoint finite sets $I_0, I_1 \subseteq \omega$, the partial type

$$\{p(x, c_i) : i \in I_0\} \cup \{q(x, c_i) : i \in I_1\}$$

is consistent. By compactness, the sequence $(c_i)_{i < \omega}$ is shattered by (p, q) . In particular, there is $d \in X/E$ such that $\text{tp}(d, c_i) = p$ iff i is even and $\text{tp}(d, c_i) = q$ iff i is odd. \square

Lemma 2.13. *Let $p, q \in S_{X/E \times Y/F}(\emptyset)$ be distinct. Then there exists an infinite sequence in Y/F shattered by (p, q) if and only if there exists an infinite sequence in X/E shattered by (p^{opp}, q^{opp}) , where $p^{opp}(x, y) := p(y, x)$.*

Proof. Let $(b_i)_{i < \omega}$ be a sequence in Y/F shattered by (p, q) . By compactness, we can find a sequence $(c_i)_{i \in \mathcal{P}(\omega)}$ in Y/F which is shattered by (p, q) as witnessed by

the family $\{d_I : I \subseteq \mathcal{P}(\omega)\} \subseteq X/E$. Consider the sequence $(d_j)_{j < \omega}$ in X/E , where $d_j := d_{I_j}$ and $I_j := \{X \subseteq \omega : j \in X\}$. Then for any $J \subseteq \omega$ we have:

$$\begin{aligned} \text{tp}(d_j, c_J) = p &\iff j \in J \\ \text{tp}(d_j, c_J) = q &\iff j \notin J, \end{aligned}$$

because $j \in J$ iff $J \in I_j$. Thus, $(d_j)_{j < \omega}$ is shattered by (p^{opp}, q^{opp}) .

The converse follows by symmetry. \square

From the last three lemmas, one easily deduces the following

Corollary 2.14. *X/E has NIP if and only if there do not exist an indiscernible sequence $(b_i)_{i < \omega}$ of elements of X/E and d (from anywhere) such that the sequence $(d, b_{2i}, b_{2i+1})_{i < \omega}$ is indiscernible and $\text{tp}(d, b_0) \neq \text{tp}(d, b_1)$.*

The next lemma is analogous to the finite-cofinite lemma on NIP theories.

Lemma 2.15. *Suppose that X/E has NIP. Let $(a_i)_{i \in I}$ be an infinite, totally indiscernible sequence of elements of X/E and b any tuple from \mathfrak{C} . Then, for any $j_0, j_1 \in I$, whenever $\text{tp}(a_{j_0}, b) \neq \text{tp}(a_{j_1}, b)$, either*

$$I_0 := \{i \in I : \text{tp}(a_i, b) = \text{tp}(a_{j_0}, b)\} \text{ is finite}$$

or

$$I_1 := \{i \in I : \text{tp}(a_i, b) = \text{tp}(a_{j_1}, b)\} \text{ is finite.}$$

Proof. Otherwise, we can build a sequence $(i_k)_{k < \omega}$ of pairwise distinct elements of I so that $i_k \in I_0$ iff k is even, and $i_k \in I_1$ iff k is odd. Then, the sequence $(a_{i_k})_{k < \omega}$ is indiscernible and

$$\begin{aligned} \text{tp}(a_{i_k}, b) = \text{tp}(a_{j_0}, b) &\iff k \text{ even} \\ \text{tp}(a_{i_k}, b) = \text{tp}(a_{j_1}, b) &\iff k \text{ odd,} \end{aligned}$$

which by Lemma 2.11 and Corollary 2.14 imply that X/E does not have NIP, a contradiction. \square

Definition 2.16. *The median value connective $\text{med}_n : [0, 1]^{2n-1} \rightarrow [0, 1]$ is defined by*

$$\text{med}_n(t_{<2n-1}) = \max_{\substack{w \subseteq \{2n-1\} \\ |w|=n}} \min_{i \in w} t_i = \min_{\substack{w \subseteq \{2n-1\} \\ |w|=n}} \max_{i \in w} t_i.$$

This connective, as its name indicates, literally computes the median value of the list of arguments.

For $f(x, y) \in \mathcal{F}_{X/E}$ and $N \in \mathbb{N}^+$, put

$$d^N f(y, x_{<2N-1}) := \text{med}_N(f(x_i, y) : i < 2N - 1).$$

As in classical model theory, generically stable types are definable, which follows from the next proposition. For $p \in S_{X/E}(\mathfrak{C})$, $f(x, y) \in \mathcal{F}_{X/E}$, and $b \in \mathfrak{C}^{|y|}$, by $f(x, b)^p$ we mean the value of p at $f(x, b)$ for p treated as an element of $S_{\mathcal{F}_{X/E}}(\mathfrak{C})$ as explained in Remark 2.5 (in other words, it is $f(a, b)$ for $a/E \models p$). We say that p is *definable* if it is so as a type in $S_{\mathcal{F}_{X/E}}(\mathfrak{C})$.

Proposition 2.17. *If $p \in S_{X/E}(\mathfrak{C})$ is generically stable over A , then for any $f(x, y) \in \mathcal{F}_{X/E}$, $b \in \mathfrak{C}^{|y|}$, and $(a_i/E)_{i < \omega}$ a Morley sequence in p over A ,*

$$|f(x, b)^p - d^{N(f, \varepsilon)} f(b, a_{<2N(f, \varepsilon)-1})| \leq \varepsilon,$$

where $N(f, \varepsilon)$ is a number as in Proposition 2.9.

Proof. Suppose this is not true. Write N for $N(f, \varepsilon)$. Then, we have two cases:

1) $d^N f(b, a_{<2N-1}) - \varepsilon > f(x, b)^p$. This implies

$$\max_{\substack{w \subseteq 2N-1 \\ |w|=N}} \min_{i \in w} f(a_i, b) > f(x, b)^p + \varepsilon.$$

Hence, there is w of size N such that for all $i \in w$ we have $f(a_i, b) > f(x, b)^p + \varepsilon$. Taking a Morley sequence in p over $Ab(a_i)_{i \in w}$, we get a contradiction with the choice of $N(f, \varepsilon)$.

2) $d^N f(b, a_{<2N-1}) + \varepsilon < f(x, b)^p$. This case is analogous to the previous, using the other definition of the median value connective. \square

Corollary 2.18. *All generically stable types in $S_{X/E}(\mathfrak{C})$ are definable.*

Proof. Consider any generically stable (over A) $p \in S_{X/E}(\mathfrak{C})$ and $f(x, y) \in \mathcal{F}_{X/E}$. Let $(a_i/E)_{i < \omega}$ be a Morley sequence in p over A . Define a CL-formula $df(y, z)$ to be the forced limit of the sequence $(d^{N(f, 2^{-n})} f(y, x_{<2N(f, 2^{-n})-1}))_{n < \omega}$ (see Definitions 3.6 and 3.8 in [BYU10]). By the last proposition and [BYU10, Lemma 3.7], we get that $df(y, (a_i)_{i < \omega})$ is the f -definition of p \square

Let $M \prec \mathfrak{C}$ (small), $f \in \mathcal{F}_{X/E}$, $p \in S_f(M)$, and $q \in S_X(\mathfrak{C})$. It is clear what it means that q extends p , namely: for $d \models q$ and for all $b \in M^{|y|}$, $f(x, b)^p = f(d, b)$ (where f is canonically extended to a bigger monster model to which d belongs). This is equivalent to saying that the partial type over M defining $\{d \in \mathfrak{C}^{|x|} : f(d, b) = f(x, b)^p \text{ for all } b \in M^{|y|}\}$ is contained in q . Thus, using the well-known fact that each partial type over M (even in infinitely many variables) extends to a global coheir over M (i.e. global type finitely satisfiable in M), we have:

Fact 2.19. *Every type $p \in S_f(M)$ has an extension to a global type $q \in S_X(\mathfrak{C})$ finitely satisfiable in M .*

Finally, we present the proof of the main result of this section.

Proof of Theorem 2.10. (1) \Leftrightarrow (2) \Leftrightarrow (3) is a part of Corollary 2.6.

(1) \Rightarrow (4) Suppose that the sequence $(a_i)_{i < \omega}$ in X/E is a counter-example. Without loss of generality we can replace ω by \mathbb{Q} . Then, there are rational numbers $i_0 < \dots < i_{n-1}$ and a natural number $j < n - 1$ such that

$$\text{tp}(a_{i_j}, a_{i_{j+1}}/A) \neq \text{tp}(a_{i_{j+1}}, a_{i_j}/A),$$

where A is the set of all a_{i_k} for $k < n$ such that $j \neq k \neq j+1$. Choose any rationals $l_0 < l_1 < \dots$ in the interval (i_j, i_{j+1}) . Let $b_i = a_{l_i}$ for $i < \omega$. Then, the sequence $(b_i)_{i < \omega}$ is A -indiscernible and $\text{tp}(b_i, b_j/A) \neq \text{tp}(b_j, b_i/A)$ for all $i < j < \omega$. This contradicts the stability of X/E .

(4) \Rightarrow (1) Assume that X/E is unstable and (4) holds. Since X/E is unstable, there exists an indiscernible sequence $(a_i/E, b_i)_{i \in \mathbb{Z}}$ such that $\text{tp}(a_i/E, b_j) \neq \text{tp}(a_j/E, b_i)$ for $i < j \in \mathbb{Z}$. The sequence $(a_i/E)_{i \in \mathbb{Z}}$ is totally indiscernible and the sets $\{i \in \mathbb{Z} : \text{tp}(a_i/E, b_0) = \text{tp}(a_1/E, b_0)\}$ and $\{i \in \mathbb{Z} : \text{tp}(a_i/E, b_0) = \text{tp}(a_{-1}/E, b_0)\}$ are both infinite, contradicting Lemma 2.15 and the assumption that X/E has NIP.

(1) \Rightarrow (5) Let $p \in S_{X/E}(\mathfrak{C})$ be invariant (over some A) but not generically stable. Then, there exists a Morley sequence $(a_i/E)_{i \in \omega + \omega}$ in p over A , $\varepsilon > 0$, $r \in \mathbb{R}$, $s \leq r - \varepsilon$, $f(x, y) \in \mathcal{F}_{X/E}$, and $b \in \mathfrak{C}^{|y|}$ such that the sets $\{i \in \omega + \omega : f(a_i, b) \leq s\}$ and $\{i \in \omega + \omega : f(a_i, b) \geq r\}$ are both infinite. There are two possible cases: either

there are infinitely many alternations, or after removing a finite number of elements a_i :

$$(f(a_i, b) \leq s \iff i < \omega) \text{ or } (f(a_i, b) \geq r \iff i < \omega).$$

By the indiscernibility of $(a_i/E)_{i \in \omega + \omega}$, Ramsey's theorem and compactness, in each case we get a contradiction with the stability of f .

(5) \Rightarrow (2) Consider any $f \in \mathcal{F}_{X/E}$ and $p \in S_f(M)$. By Fact 2.19, choose $q' \in S_X(\mathfrak{C})$ extending p which is a coheir over M ; let $q \in S_{X/E}(\mathfrak{C})$ be induced by q' . Being a coheir over M , q' is M -invariant; so q is M -invariant, hence generically stable by (5). By Corollary 2.18, q is definable. Denote the f -definition of q by ψ . Since q is M -invariant, so is the CL-formula ψ . Therefore, ψ is definable over M (i.e. a CL-formula over M). Hence, p is definable by the CL-formula ψ . \square

3. DISTAL HYPERIMAGINARY SEQUENCES

The goal of this section is to prove Proposition 1.3 and deduce Proposition 1.2, which in turn confirms the prediction from [HP18] that in a distal theory $G^{st} = G^{00}$.

We work in a monster model \mathfrak{C} of a complete, first order theory T with NIP. This section is based on [Sim13], in particular the next two definitions are from there.

Definition 3.1. For any indiscernible sequence I , if $I = I_1 + I_2$ (the concatenation of I_1 and I_2), we say that $\mathfrak{c} = (I_1, I_2)$ is a cut of I .

We write $(I'_1, I'_2) \trianglelefteq (I_1, I_2)$ if I'_1 is an end segment of I_1 and I'_2 an initial segment of I_2 .

If $J \subset I$ is a convex subsequence, a cut $\mathfrak{c} = (I_1, I_2)$ is said to be interior to J if $I_1 \cap J$ and $I_2 \cap J$ are infinite.

A cut is Dedekind if both I_1 and I_2^* (I_2 with the reversed order) have infinite cofinality.

A polarized cut is a pair $(\mathfrak{c}, \varepsilon)$, where \mathfrak{c} is a cut (I_1, I_2) and $\varepsilon \in \{1, 2\}$ is such that I_ε is infinite.

If $\mathfrak{c} = (I_1, I_2)$ is a cut, we say that a tuple b fills \mathfrak{c} if $I_1 + b + I_2$ is indiscernible.

Sometimes, if it is clear that the tuple b fills some cut $\mathfrak{c} = (I_1, I_2)$ of I , we will write $I \cup \{b\}$ instead of $I_1 + b + I_2$. And similarly, in the case of two elements a, b filling respectively distinct cuts $\mathfrak{c}_1, \mathfrak{c}_2$, abusing notation, we will write $I \cup \{a\} \cup \{b\}$ for the associated concatenation.

Definition 3.2. A dense indiscernible sequence I is distal if for any distinct Dedekind cuts $\mathfrak{c}_1, \mathfrak{c}_2$, if a fills \mathfrak{c}_1 and b fills \mathfrak{c}_2 , then $I \cup \{a\} \cup \{b\}$ is indiscernible.

The theory T is distal if all dense indiscernible sequences (of tuples from the home sort) are distal.

We say that T^{heq} is distal if all dense indiscernible sequences $(a_i/E)_{i \in \mathcal{I}}$ of hyperimaginaries (where E is \emptyset - \wedge -definable) are distal.

Let E be a \emptyset - \wedge -definable equivalence relation on \mathfrak{C}^λ , and let $\pi_E : \mathfrak{C}^\lambda \rightarrow \mathfrak{C}^\lambda/E$ be the quotient map.

The next lemma is a variant of [Sim13, Lemma 2.8] for hyperimaginaries.

Lemma 3.3. Let $I = (a_i/E)_{i \in \mathcal{I}}$ be a dense indiscernible sequence and $A \subset \mathfrak{C}$ a (small) set of parameters. Let $(\mathfrak{c}_i)_{i < \alpha}$ be a sequence of pairwise distinct Dedekind cuts in I . For each $i < \alpha$ let d_i fill the cut \mathfrak{c}_i . Fix a polarization of each \mathfrak{c}_i , $i < \alpha$.

Then there are $(d'_i)_{i < \alpha}$ satisfying $(d'_i)_{i < \alpha} \equiv_I (d_i)_{i < \alpha}$ such that for every formula θ with parameters from A and $i < \alpha$: if

$$\pi_E^{-1}(d'_i) \subseteq \theta(\mathfrak{C}),$$

then

$$\pi_E^{-1}(a_j/E) \not\subseteq \neg\theta(\mathfrak{C})$$

for a_j/E from a co-final fragment of the left part of \mathfrak{c}_i if \mathfrak{c}_i is left-polarized, or from a co-initial fragment of the right part of \mathfrak{c}_i if \mathfrak{c}_i is right-polarized.

Proof. To simplify notation, let all the cuts \mathfrak{c}_i be left-polarized. The negation of the conclusion says that for every $(d'_i)_{i < \alpha} \equiv_I (d_i)$ there exists $i < \alpha$ and a formula $\theta(x)$ over A such that

$$\pi_E^{-1}(d'_i) \subseteq \theta(\mathfrak{C})$$

and

$$\pi_E^{-1}(a_j/E) \subseteq \neg\theta(\mathfrak{C})$$

for all a_j/E in some end segment of \mathfrak{c}_i . For $i < \alpha$ put $C_i(x_i) := \{\varphi(x_i) \in L(A) : \pi_E^{-1}(a_j/E) \subseteq \varphi(\mathfrak{C}) \text{ for all } a_j/E \text{ in some end segment of } \mathfrak{c}_i\}$. Note that these sets are closed under conjunction. The negation of the conclusion is equivalent to

$$(*) \quad \text{tp}((d_i)_{i < \alpha}/I) \cup \bigcup_{i < \alpha} C_i(x_i) \text{ is inconsistent,}$$

which in turn is equivalent to the existence of a finite subset J of α such that $\text{tp}((d_i)_{i \in J}/I) \cup \bigcup_{i \in J} C_i(x_i)$ is inconsistent. Therefore, without loss of generality, $\alpha < \omega$. We will show a detailed proof for $\alpha = 2$; the proof for an arbitrary $\alpha \in \omega$ is the same.

Suppose the conclusion fails. By (*), choose a finite subsequence I_0 of I so that $\text{tp}(d_0, d_1/I_0) \cup C_0(x_0) \cup C_1(x_1)$ is inconsistent, and let $\varphi_0(x_0) \in C_0(x_0)$ and $\varphi_1(x_1) \in C_1(x_1)$ be formulas witnessing it. Now, for $i \in \{0, 1\}$ take $(J_i, J'_i) \trianglelefteq \mathfrak{c}_i$ such that $\pi_E^{-1}(a_j/E) \subseteq \varphi_i(\mathfrak{C})$ for all $a_j/E \in J_i$, $J_i \cup J'_i$ contains no element of I_0 , and $(J_0 \cup J'_0) \cap (J_1 \cup J'_1) = \emptyset$.

Claim. For every two cuts $\mathfrak{d}_0, \mathfrak{d}_1$ in I respectively interior to J_0, J_1 we can find hyperimaginaries e_0 filling \mathfrak{d}_0 and e_1 filling \mathfrak{d}_1 such that $\text{tp}(e_0, e_1/I_0) = \text{tp}(d_0, d_1/I_0)$.

Proof of claim. Consider any finite $K \subseteq I$. For $i \in \{0, 1\}$, the cut \mathfrak{d}_i decomposes K into $L_i^- + L_i^+$. It is enough to find e_0, e_1 such that

$$\begin{aligned} \text{tp}(L_1^-, e_0, L_1^+) &\subset \text{EM}(I), \\ \text{tp}(L_2^-, e_1, L_2^+) &\subset \text{EM}(I), \\ \text{tp}(e_0, e_1/I_0) &= \text{tp}(d_0, d_1/I_0), \end{aligned}$$

where $\text{EM}(I)$ denotes the Ehrenfeucht-Mostowski type of I .

We can decompose K into sequences $K_0 \subseteq J_0 + J'_0$, $K_1 \subseteq J_1 + J'_1$, and $K_2 \subseteq I \setminus ((J_0 + J'_0) \cup (J_1 + J'_1))$. Next, we construct new finite sequences K'_0, K'_1, K'_2 , and K' in the following way: $K'_2 = K_2$; for every element $a \in K_0$ we take $a' \in J_0 \cup J'_0$ such that a' is in the same relative position to \mathfrak{c}_0 as a was to \mathfrak{d}_0 and also preserving the order between elements, and we define K'_0 to be the constructed sequence of the a' 's; for K'_1 we proceed in an analogous manner; finally, $K' := K'_0 \cup K'_1 \cup K'_2$ written as a sequence in an obvious order provided by the construction. By the indiscernibility of the sequence I , there is $\sigma \in \text{Aut}(\mathfrak{C})$ such that $\sigma(K'I_0) = K'I_0$. The elements $e_0 := \sigma^{-1}(d_0)$ and $e_1 := \sigma^{-1}(d_1)$ satisfy the desired conditions. \square

Fix $\mathfrak{d}_0, \mathfrak{d}_1$ as in the claim, and choose $e_0 =: e_0^0$ and $e_1 =: e_1^0$ provided by the claim. By the choice of $\varphi_i(x)$, $\pi_E^{-1}(e_0^0) \subseteq \neg\varphi_0(\mathfrak{C})$ or $\pi_E^{-1}(e_1^0) \subseteq \neg\varphi_1(\mathfrak{C})$. For example, $\pi_E^{-1}(e_0^0) \subseteq \neg\varphi_0(\mathfrak{C})$. Set $I^0 := I \cup \{e_0^0\}$; this is an indiscernible sequence. Let J_0^0 be an end segment of J_0 not containing \mathfrak{d}_0 , and $J_1^0 := J_1$. By the same argument as in the above claim, we get

Claim. *For every two cuts $\mathfrak{d}_0^1, \mathfrak{d}_1^1$ in I^0 respectively interior to J_0^0, J_1^0 we can find hyperimaginaries e_0^1 filling \mathfrak{d}_0^1 and e_1^1 filling \mathfrak{d}_1^1 (seen as cuts in I^0) such that $\text{tp}(e_0^1, e_1^1/I_0) = \text{tp}(d_0, d_1/I_0)$.*

Fix $\mathfrak{d}_0^1, \mathfrak{d}_1^1$ as in the claim, and choose e_0^1, e_1^1 provided by the claim. Then $\pi_E^{-1}(e_0^1) \subseteq \neg\varphi_0(\mathfrak{C})$ or $\pi_E^{-1}(e_1^1) \subseteq \neg\varphi_1(\mathfrak{C})$. For example, $\pi_E^{-1}(e_0^1) \subseteq \neg\varphi_0(\mathfrak{C})$. Set $I^1 := I^0 \cup \{e_0^1\}$; this is again an indiscernible sequence. Let $J_0^1 := J_0^0$, and J_1^1 be an end segment of J_1^0 not containing \mathfrak{d}_1^1 .

Iterating this process ω times, we get a sequence $(\varepsilon_k)_{k < \omega}$ of 0's and 1's and a sequence of hyperimaginaries $(e_{\varepsilon_k}^k)_{k < \omega}$ such that $I \cup \{e_{\varepsilon_k}^k\}_{k < \omega}$ is indiscernible, the $e_{\varepsilon_k}^k$'s with $\varepsilon_k = 0$ fill pairwise distinct cuts in J_0 , the $e_{\varepsilon_k}^k$'s with $\varepsilon_k = 1$ fill pairwise distinct cuts in J_1 , and $\pi_E^{-1}(e_{\varepsilon_k}^k) \subseteq \neg\varphi_{\varepsilon_k}(\mathfrak{C})$ for all $k < \omega$. W.l.o.g. $\varepsilon_k = 0$ for all $k < \omega$.

Finally, by Ramsey's theorem and compactness, we can find an indiscernible sequence of representatives of the hyperimaginaries from the indiscernible sequence $J_0 \cup \{e_0^k\}_{k < \omega}$. In this way, we have produced an indiscernible sequence for which $\varphi_0(x_0)$ has infinite alternation rank, which contradicts NIP. \square

Theorem 3.4. *If $(a_i)_{i \in \mathcal{I}}$ is a (dense) distal sequence of tuples from \mathfrak{C}^λ , then $(a_i/E)_{i \in \mathcal{I}}$ is a distal sequence of hyperimaginaries. Thus, if T is distal, then T^{heq} is distal.*

Proof. The fact that the first part implies the second follows from the observation that for any indiscernible sequence of hyperimaginaries we can find an indiscernible sequence of representatives. So let us prove the first part.

Assume $I := (a_i)_{i \in \mathcal{I}}$ is a (dense) distal sequence of tuples from \mathfrak{C}^λ , and let $I' = (a_i/E)_{i \in \mathcal{I}}$. Present E as $\bigcap_{t \in \mathcal{T}} R_t$ for some \emptyset -definable (not necessarily equivalence) relations R_t . Consider any distinct Dedekind cuts \mathfrak{c}'_1 and \mathfrak{c}'_2 of I' , say \mathfrak{c}'_1 is on the left from \mathfrak{c}'_2 . They partition I' into I'_1, I'_2 , and I'_3 . The cuts \mathfrak{c}'_1 and \mathfrak{c}'_2 induce Dedekind cuts \mathfrak{c}_1 and \mathfrak{c}_2 of I which partition I into I_1, I_2 , and I_3 . Take any d_1 and d_2 filling the cuts \mathfrak{c}'_1 and \mathfrak{c}'_2 , respectively. Apply Lemma 3.3 to this data (taking left-polarization of \mathfrak{c}'_1 and \mathfrak{c}'_2) and A being the set of all coordinates of all tuples a_i , $i \in \mathcal{I}$. This yields $d'_1 = e'_1/E$ and $d'_2 = e'_2/E$ satisfying the conclusion of Lemma 3.3.

Claim. *For every $i \in \{1, 2\}$ there is b_i filling the cut \mathfrak{c}_i such that $\pi_E(b_i) = d'_i$.*

Proof of the claim. It is enough to consider $i = 1$. By compactness, it suffices to show that for any formula $\varphi(x_1, \dots, x_n) \in \text{EM}(I)$, $i_1 < \dots < i_{k-1} \in \mathcal{I}_1$, and $i_{k+1} < \dots < i_n \in \mathcal{I}_2 + \mathcal{I}_3$ there is b_1 such that $\models \varphi(a_{i_1}, \dots, a_{i_{k-1}}, b_1, a_{i_{k+1}}, \dots, a_{i_n})$ and $\pi_E(b_1) = d'_1$. By compactness, it is enough to show that for every $t \in \mathcal{T}$

$$\models \exists y (y R_t e'_1 \wedge \varphi(a_{i_1}, \dots, a_{i_{k-1}}, y, a_{i_{k+1}}, \dots, a_{i_n})).$$

Assume this fails. Choosing $t' \in \mathcal{T}$ such that $R_{t'} \circ R_{t'} \subseteq R_t$, we get

$$\pi_E^{-1}(d'_1) \subseteq \neg(\exists y)(y R_{t'} x \wedge \varphi(a_{i_1}, \dots, a_{i_{k-1}}, y, a_{i_{k+1}}, \dots, a_{i_n}))(\mathfrak{C}) =: \theta(\mathfrak{C}).$$

By the choice of d'_1 , for a_j/E from a co-final fragment of the left part of \mathfrak{c}_1 we have $\pi_E^{-1}(a_j/E) \not\subseteq \neg\theta(\mathfrak{C})$. So, for any of these indices j , there is $a'Ea_j$ such that

$$\models \neg\exists y(yR_{t'}a' \wedge \varphi(a_{i_1}, \dots, a_{i_{k-1}}, y, a_{i_{k+1}}, \dots, a_{i_n})).$$

As the set of such indices j is co-final in \mathcal{I}_1 , we can find such an index $j \in \mathcal{I}_1$ with $j > i_{k-1}$. Then $y := a_j$ contradicts the last formula (by the indiscernibility of I). \square

By the distality of I , the sequence $I_1 + b_1 + I_2 + b_2 + I_3$ is indiscernible. Hence, the sequence $I'_1 + d'_1 + I'_2 + d'_2 + I'_3$ is also indiscernible. On the other hand, by our choice of d'_1, d'_2 , we know that $d_1 d_2 \equiv_{I'} d'_1 d'_2$. Thus, the sequence $I'_1 + d_1 + I'_2 + d_2 + I'_3$ is indiscernible, too. As d_1, d_2 were arbitrary, we conclude that I' is a distal sequence. \square

Corollary 3.5. *For a distal theory T , a hyperdefinable set X/E is stable if and only if E is a bounded equivalence relation. In particular, for a group $G \wedge$ -definable in a distal theory, $G^{st} = G^{00}$.*

Proof. If E is bounded, then X/E is stable (as each indiscernible sequence in X/E is constant). To prove the other implication, assume that X/E is stable. Since distality is preserved under naming parameters, w.l.o.g. both X and E are \emptyset -type-definable. If X/E is not bounded, taking a very long sequence of pairwise distinct elements of X/E , by extracting indiscernibles, there exists a dense indiscernible sequence of pairwise distinct elements of X/E . By stability and Theorem 2.10, this sequence is totally indiscernible. Since non-constant, totally indiscernible sequences are not distal, we get a contradiction with the distality of T^{heq} (which we have by Theorem 3.4). \square

One could give a short direct (i.e. not using Theorem 3.4) proof of Corollary 3.5. However, we find it very natural to see Corollary 3.5 as an easy consequence of Theorem 3.4 which in turn is a fundamental result concerning distality and hyperimaginaries.

4. AN EXAMPLE OF $G^{st,0} \neq G^{st} \neq G^{00} = G$

Our objective is to find a definable group G in a NIP theory T satisfying $G^{st,0} \neq G^{st} \neq G^{00} = G$. In this section, we will change the notation: the group interpreted in the monster model will be denoted by G^* instead of G .

Consider the structure $\mathcal{M} := (\mathbb{R}, +, I)$, where $I := [0, 1]$. Let $T := \text{Th}(\mathcal{M})$ and $G := (\mathbb{R}, +)$. Let $\mathcal{M}^* = (\mathbb{R}^*, +, I^*) \succ \mathcal{M}$ be a monster model (κ -saturated and strongly κ -homogeneous for large κ) which expands to a monster model $(\mathbb{R}^*, +, \leq, 1) \succ (\mathbb{R}, +, \leq, 1)$ (with the same κ), and $G^* := (\mathbb{R}^*, +)$. Denote by μ the subgroup of infinitesimals, i.e. $\bigcap_{n \in \mathbb{N}^+} [-1/n, 1/n]^*$.

Some observations below may follow from more general statements in the literature, but we want to be self-contained and as elementary as possible in the analysis of this example.

Proposition 4.1. *T has NIP and is unstable.*

Proof. The structure \mathcal{M} is the reduct of the o-minimal structure $(\mathbb{R}, +, \leq, 1)$, hence T has NIP.

Note that for $\varepsilon \in (0, \frac{1}{2})$ we can write the interval $[-\varepsilon, \varepsilon]$ as $(I - \varepsilon) \cap (I + (\varepsilon - 1))$. Hence, the formula $\varphi(x, y) := "x \in I - y \wedge x \in I + (y - 1)"$ has SOP. \square

Remark 4.2. $0, 1 \in \text{dcl}^{\mathcal{M}}(\emptyset)$.

Proof. $0 \in \text{dcl}^{\mathcal{M}}(\emptyset)$ as the neutral element of $(\mathbb{R}, +)$. To see that $1 \in \text{dcl}^{\mathcal{M}}(\emptyset)$, note that 1 is defined by the formula $\varphi(x) := "x \in I \wedge \forall y(y \in I \implies (x - y) \in I)"$. \square

Lemma 4.3. (1) *The only invariant subgroups of G^* are: $\{0\}$, the subgroups of \mathbb{Q} , the subgroups of the form $\mu + R$ where R is a subgroup of \mathbb{R} , and G^* .*
 (2) *The only \emptyset - \wedge -definable subgroups of G^* are $\{0\}$, μ , and G^* .*
 (3) *The only definable (over parameters) subgroups of G^* are $\{0\}$ and G^* .*

Proof. (1) By q.e. in $(\mathbb{R}, +, \leq, 1)^*$, Remark 4.2, and the fact that the order restricted to any interval $[-r, r]$ (where $r \in \mathbb{N}^+$) is \emptyset -definable in \mathcal{M} (see Lemma 4.7), the following holds in \mathcal{M}^* :

- (i) All elements $a > \mathbb{R}$ have the same type over \emptyset .
- (ii) $\text{dcl}^{\mathcal{M}}(\emptyset) = \mathbb{Q}$.
- (iii) For any $a \in \mathbb{R} \setminus \mathbb{Q}$, $a + \mu$ is the set of all realizations of a type in $S_1(\emptyset)$.
- (iv) For any $a \in \mathbb{Q}$, all the elements $a + h$, where h ranges over positive infinitesimals, form the set of realizations of a type in $S_1(\emptyset)$; and the same is true for all elements $a - h$.

This easily implies that the groups in the lemma are indeed invariant.

For the converse, let H be an invariant subgroup. If it contains some $a > \mathbb{R}$ or $a < \mathbb{R}$, then $H = G^*$ by (i). So suppose that $H \subseteq \mathbb{R} + \mu$. If H contains an element from $a + \mu$ for some $a \in \mathbb{R} \setminus \mathbb{Q}$, then, by (iii), it contains $a + \mu$ and so μ as well; thus, H is of the form $\mu + R$, where R is a subgroup of \mathbb{R} . If H contains some element $a + h$ with $a \in \mathbb{Q}$ and h a positive infinitesimal, then, by (iv), H contains all the elements of that form. Since H is a group, it contains μ (because we can subtract any two elements $a + h, a + h'$); thus, H is again of the form $\mu + R$, where R is a subgroup of \mathbb{R} . If H contains some element of the form $a - h$ with $a \in \mathbb{Q}$ and h a positive infinitesimal, we proceed in an analogous manner. The only remaining case is that H is a subgroup of \mathbb{Q} .

(2) A \emptyset - \wedge -definable subgroup H either contains some $a > \mathbb{R}$, in which case $H = G^*$, or the type defining H implies the formula $x \leq n$ for some $n \in \mathbb{N}$. This implies that H is contained in μ , so, by (1), either $H = \{0\}$ or $H = \mu$.

(3) By o-minimality of $(\mathbb{R}, +, \leq, 1)^*$, any definable subgroup (over parameters) H of G^* is a finite union of points and intervals, so the conclusion easily follows. \square

Corollary 4.4. $G^* = G^{*0} = G^{*00} = G^{*000}$

Proof. Since $G^* \geq G^{*0} \geq G^{*00} \geq G^{*000}$, it is enough to show that $G^{*000} = G^*$. But this follows from Lemma 4.3(1), as the index of $\mathbb{R} + \mu$ in G^* is unbounded. \square

Corollary 4.5. $G^{*st,0} = G^*$

Proof. It follows directly from Lemma 4.3(3) and Proposition 4.1. \square

Let $\mathcal{N} := (\mathbb{R}, +, -, R_r)_{r \in \mathbb{N}^+}$, where $R_r(x, y)$ holds if and only if $0 \leq y - x \leq r$. Let $T' := \text{Th}(\mathcal{N})$.

Remark 4.6. *The family $(R_r)_{r \in \mathbb{N}^+}$ satisfy the following conditions:*

- (1) $R_r(x, y) \iff R_r(0, y - x)$;
- (2) $R_r(x, y) \iff R_r(-y, -x)$;
- (3) $R_r(x, y) \iff R_{nr}(nx, ny)$ (where $n \in \mathbb{N}^+$);

$$(4) R_r(x, y) \iff R_1(x, y) \vee R_1(x, y-1) \vee \cdots \vee R_1(x, y-(r-1)).$$

Lemma 4.7. *The structures \mathcal{M} and \mathcal{N} are interdefinable over \emptyset .*

Proof. \mathcal{M} is definable over \emptyset in \mathcal{N} , because $x \in [0, 1]$ iff $R_1(0, x)$ iff $R_1(x, 2x)$.

To see that \mathcal{N} is definable over $\{1\}$ in \mathcal{M} , note that the function $-$ can be defined using $+$ as usual, $R_1(x, y) \iff y - x \in [0, 1]$, and then use the last property in Remark 4.6 to conclude that all R_r , $r \in \mathbb{N}^+$, are definable over $\{1\}$ in \mathcal{M} . Since $1 \in \text{dcl}^{\mathcal{M}}(\emptyset)$, we conclude that \mathcal{N} is definable over \emptyset in \mathcal{M} . \square

The result above shows that the theory T' also has NIP and is unstable.

Proposition 4.8. *T' has quantifier elimination after expansion by the constant 1.*

Proof. We argue by induction on the length of the formula. So the proof boils down to showing that a primitive formula $(\exists y)\varphi(y, \bar{x})$ is T' -equivalent to a quantifier free formula, assuming that all shorter formulas are T' -equivalent to qf-formulas. Recall that $(\exists y)\varphi(y, \bar{x})$ being primitive means that $\varphi(y, \bar{x})$ is a conjunction of atomic formulas and negations of such, i.e. $\varphi(y, \bar{x}) = \bigwedge_{j=1}^m R_{r_j}^{\varepsilon_j}(t_j^l(y, \bar{x}), t_j^r(y, \bar{x}))$, where $\varepsilon_j \in \{\pm 1, \pm 2\}$, $r_j \in \mathbb{N}^+$, $t_j^l(y, \bar{x})$ and $t_j^r(y, \bar{x})$ are terms, and: $R_{r_j}^{-2}(t, z) := \neg R_{r_j}(t, z)$, $R_{r_j}^2(t, z) := R_{r_j}(t, z)$, $R_{r_j}^{-1}(t, z) := \neg(t = z)$, $R_{r_j}^1(t, z) := (t = z)$.

Using $+$, $-$, multiplying by suitable integers, Remark 4.6, and induction hypothesis, we can assume that there is an integer $n \neq 0$ such that for every j : either $t_j^l(y, \bar{x}) = 0$ and $t_j^r(y, \bar{x}) = ny - t_j(\bar{x})$, or $t_j^l(y, \bar{x}) = ny - t_j(\bar{x})$ and $t_j^r(y, \bar{x}) = 0$.

If some $\varepsilon_j = 1$, one gets $ny = t_j(\bar{x})$ and the quantifier $\exists y$ can be eliminated. So assume that all $\varepsilon_j \neq 1$. If additionally all $\varepsilon_j \neq 2$, then $(\exists y)\varphi(y, \bar{x})$ is T' -equivalent to \top . So assume that some $\varepsilon_j = 2$, e.g. $\varepsilon_1 = 2$. Then $R_{r_1}^{\varepsilon_1}(t_1^l(y, \bar{x}), t_1^r(y, \bar{x}))$ either says (in \mathcal{N}) that $ny \in [t_1(\bar{x}) - r_1, t_1(\bar{x})]$, or that $ny \in [t_1(\bar{x}), t_1(\bar{x}) + r_1]$. Suppose the latter case holds. Consider all (finitely many) possibilities taking into account:

- which terms $t_j(\bar{x}) - r_j, t_j(\bar{x}), t_j(\bar{x}) + r_j$ (for $j \in \{2, \dots, m\}$) belong to $[t_1(\bar{x}) - r_1, t_1(\bar{x})]$;
- for those which belong to this interval, how they are ordered by R_{r_1} ;
- for $\varepsilon_j \neq -1$, writing $R_{r_j}^{\varepsilon_j}(t_j^l(y, \bar{x}), t_j^r(y, \bar{x}))$ as “ $ny \in I$ ” or as “ $ny \notin I$ ”, where $I := [t_j(\bar{x}) - r_j, t_j(\bar{x})]$ or $I := [t_j(\bar{x}), t_j(\bar{x}) + r_j]$, we should specify which of the terms $t_1(\bar{x})$ and $t_1(\bar{x}) + r_1$ belong to I .

Each of these possibilities is clearly a qf-definable condition on \bar{x} (using finitely many integers, but they are terms, as 1 was added to the language). On the other hand, $(\exists y)\varphi(y, \bar{x})$ is T' -equivalent to the disjunction of some subfamily of these conditions (by a simple combinatorics on intervals). Therefore, $(\exists y)\varphi(y, \bar{x})$ is T' -equivalent to a qf-formula. \square

Proposition 4.9. *The quotient G^*/μ is stable.*

Proof. By Theorem 2.10, it is enough to show that for every $A \subseteq \mathcal{M}^*$ with $|A| \leq \mathfrak{c}$ we have $|S_{G^*/\mu}(A)| \leq \mathfrak{c}$.

By Lemma 4.7, \mathcal{M}^* can be treated as an elementary extension of \mathcal{N} .

Consider an arbitrary set A as above. Put $V_A := \text{Lin}_{\mathbb{Q}}(A \cup \mathbb{R})$ and $\tilde{V}_A := V_A + \mu$. First, note that for any $a, a' \in G^*$, if $a - a' \in \mu$, then trivially $\text{tp}((a + \mu)/A) = \text{tp}((a' + \mu)/A)$. Now, consider any $a \in G^* \setminus \tilde{V}_A$. Then a satisfies the formulas

$$kx \neq t$$

and

$$\neg R_r(t, kx)$$

for all $k \in \mathbb{Z} \setminus \{0\}$, $r \in \mathbb{N}^+$, and $t \in V_A$. By Remark 4.6, Lemma 4.7 and Proposition 4.8, these formulas completely determine $\text{tp}(a/V_A)$. Hence, any $a, a' \in G^* \setminus \tilde{V}_A$ have the same type over A . Therefore, $|S_{G^*/\mu}(A)| \leq |\tilde{V}_A/\mu| + 1 \leq |V_A| + 1 = \mathfrak{c}$. \square

Corollary 4.10. $\mu = G^{*st}$.

Proof. It follows from Lemma 4.3(2), Proposition 4.9, and Proposition 4.1. \square

To summarize, we have proved that G^* is a \emptyset -definable group in a monster model of a NIP theory such that $G^{*st,0} \neq G^{*st} \neq G^{*00} = G^*$.

5. HOW TO CONSTRUCT EXAMPLES WITH $G^{st} \neq G^{st,0}$?

Our context is that G is a \emptyset -definable group in a monster model \mathfrak{C} of a complete theory T with NIP.

Proposition 5.1. *If $G^{st} = G^{st,0}$, then any \wedge -definable subgroup H of G with G/H stable is an intersection of definable subgroups.*

Proof. Assume that $G^{st} = G^{st,0} = \bigcap_{j \in J} G_j$, where G_j are definable groups. Since $G/G^{st,0}$ is a stable hyperdefinable group, by [HP18, Remark 2.5(iv)], the intersection of all the conjugates of each G_j is a bounded subintersection, and so it is a finite subintersection (by compactness and definability of G_j). Replacing each G_j by such a finite subintersection, we can assume that all the G_j 's are normal subgroups of G . Hence, for every $j \in J$ we have $G_j \leq H \cdot G_j \leq G$ and

$$H \cdot G_j / G_j \leq G / G_j.$$

Using Hrushovski's theorem (see [Pil96, Ch. 1, Lemma 6.18]) inside the (definable) stable group G/G_j , we get

$$H \cdot G_j / G_j = \bigcap_{i \in I_j} K_j^i / G_j,$$

for some definable subgroups K_j^i of G such that $K_j^i \cdot G_j = K_j^i$ for all $i \in I_j$.

Since G/H is assumed to be stable, $G^{st} \leq H$. Thus,

$$H = H \cdot \bigcap_{j \in J} G_j = \bigcap_{j \in J} H \cdot G_j = \bigcap_{j \in J} \bigcap_{i \in I_j} K_j^i.$$

\square

If we do not require that $G^{00} = G^0$, then it is easy to find examples where $G^{00} \neq G^{st} \neq G^{st,0}$; that is why [HP18] required in this problem also $G^{00} = G$. However, the requirement $G^{00} = G^0$ seems sufficiently interesting. The next proposition yields the whole class of examples where $G^{00} \neq G^{st} \neq G^{st,0}$ (but without the requirement that $G^{00} = G^0$).

Proposition 5.2. *Let G be definably isomorphic to a definable semidirect product of definable groups H and K (symbolically, $G \cong_{\text{def}} H \rtimes K$) such that $H^{00} \neq H^0$ and $K^{st} \neq K^{00}$. Then $G^0 \neq G^{00} \neq G^{st} \neq G^{st,0}$.*

Proof. W.l.o.g. $G = H \times K$ and H , K , and the action of H on K are \emptyset -definable. Recall that by NIP, the 00-components exist (i.e. do not depend on the choice of parameters over which they are computed). Hence, K^{00} is invariant under all definable automorphisms, in particular under the action of H . So $G^{00} = H^{00} \times K^{00}$ (e.g. by Corollary 4.11 in [GJK22]). But $G^{st} \leq H^{00} \times K^{st}$, because the map $(h, k)/(H^{00} \times K^{st}) \mapsto h/H^{00} \times k/K^{st}$ is an invariant bijection from $G/(H^{00} \times K^{st})$ to $H/H^{00} \times K/K^{st}$ and the last set is stable as a product of stable sets. Thus, since K^{st} is a proper subgroup of K^{00} , we get that $G^{00} \neq G^{st}$.

To see that $G^{st} \neq G^{st,0}$, it is enough to note that $H^{00} \times K \leq G$ and that $H^{00} \times K$ is not an intersection of definable groups (because $G/(H^{00} \times K) \cong_{def} H/H^{00}$ is not profinite in the logic topology as $H^{00} \neq H^0$). Indeed, having this, since $G/(H^{00} \times K)$ is bounded and so stable, by Proposition 5.1, we conclude that $G^{st} \neq G^{st,0}$.

The fact that $G^{00} \neq G^0$ follows from $H^{00} \neq H^0$, as $G^0 = H^0 \times K^0$. \square

Remark 5.3. *The assumption of Proposition 5.2 is equivalent to saying that G has a definable, normal subgroup K with $K^{st} \neq K^{00}$ and $(G/K)^{00} \neq (G/K)^0$ such that the quotient map $G \rightarrow G/K$ has a section which is a definable homomorphism.*

The proof of Proposition 5.2 can be easily modified to get the following variant.

Remark 5.4. *The conclusion of Proposition 5.2 remains true with the assumption “ $H^{00} \neq H^0$ and $K^{st} \neq K^{00}$ ” replaced by “ $H^{st} \neq H^0$ and $K^{00} \neq K^0$ ”.*

One can find many examples satisfying the assumptions of Proposition 5.2. For instance, take any group H (definable in a monster model of a NIP theory T_1) with $H^{00} \neq H^0$ (e.g. the circle group in the theory of real closed fields) and any group K (definable in a monster model of a NIP theory T_2 ; where T_1 and T_2 are in disjoint languages) with $K^{st} \neq K^{00}$ (e.g. T_2 is stable and K is infinite). Consider T being the union of T_1 and T_2 living on two disjoint sorts. Then $G := H \times K$ satisfies the assumptions of Proposition 5.2 as a group definable in T .

One could still ask if it is possible to find examples satisfying the condition $G^0 = G^{00} \neq G^{st} \neq G^{st,0}$ by finding a definable, normal subgroup K satisfying $K^{00} \neq K^0$ where $G \rightarrow G/K$ does not have a definable section. However, there is no chance for this potential method to work for groups of finite exponent, as for any torsion (equivalently finite exponent) group K definable in a monster model, we have $K^{00} = K^0$. This is because K/K^{00} is a compact torsion group, and such groups are known to be profinite (see [HR13, Theorem 8.20]).

Problem 5.5. *Construct G of finite exponent with $G^{00} \neq G^{st} \neq G^{st,0}$. (The equality $G^0 = G^{00}$ always holds by the fact at the end of the last paragraph.)*

In the final part of this section, we describe how one could try to construct examples where $G^{00} \neq G^{st} \neq G^{st,0}$. In fact, originally we used this approach to find the example in Section 4. We will also point out a difference between the situation in the example from Section 4 and the finite exponent case.

Proposition 5.6. *The conditions $G^{st,0} \neq G^{st}$ and $G^{st} \neq G^{00}$ are equivalent to the existence of a \wedge -definable subgroup H of G such that:*

- (1) H is a countable intersection $\bigcap_{n < \omega} D_n$ of definable subsets of G satisfying $D_{n+1} D_{n+1} \subseteq D_n$ and symmetric (i.e. $D_n^{-1} = D_n$ and $e \in D_n$);
- (2) $[G : H]$ is unbounded;
- (3) H is not an intersection of definable subgroups;

(4) G/H is stable.

Proof. If $G^{st} \neq G^{00}$, then, by compactness, $G^{st} = \bigcap \{H : H \text{ satisfies (1), (2), (4)}\}$. Hence, assuming additionally that $G^{st,0} \neq G^{st}$, at least one of those groups H has to also satisfy condition (3).

Assume now that $H \leq G$ satisfies conditions (1), (2), (3), and (4). Since $G^{st} \leq H$, we get $G^{st} \neq G^{00}$. The fact that $G^{st,0} \neq G^{st}$ follows from Proposition 5.1. \square

Note that assuming (1), the negation of (3) is equivalent to saying that for every $n < \omega$ there is $m > n$ and a definable subgroup K of G such that $D_m \subseteq H \subseteq D_n$.

Remark 5.7. *If we have a situation as in the last proposition, then the same holds for G treated as a group definable in $(G, \cdot, (D_n)_{n < \omega})$.*

So an idea is to look for a group G and a decreasing sequence $(D_n)_{n < \omega}$ of symmetric subsets of G with $D_{n+1}D_{n+1} \subseteq D_n$ for all $i < \omega$, such that for $M := (G, \cdot, (D_n)_{n < \omega})$ and $G^* := G(\mathfrak{C})$ (where $\mathfrak{C} = (G^*, +, (D_n^*)_{n < \omega}) \succ M$ is a monster model), the group $H := \bigcap_{n < \omega} D_n^*$ satisfies (1)-(4) from the last proposition (with $*$ added everywhere). In the example from Section 4, $G := (\mathbb{R}, +)$ and as D_n we can take $[-1/2^n, 1/2^n]$. Then the D_n 's are definable in $(\mathbb{R}, +, [0, 1])$, hence M is interdefinable with $(\mathbb{R}, +, [0, 1])$, and so we focused on the latter structure. In the proof of stability of G^*/μ (see Proposition 4.9), for the counting argument to work it was important that D_{n+1} is generic in D_n (i.e. finitely many translates of D_{n+1} cover D_n), as this guarantees that $\mu = \bigcap D_n^*$ has bounded index in the subgroup generated by D_1^* . The next proposition shows that for abelian groups of finite exponent this genericity condition always fails.

Proposition 5.8. *If G is abelian of finite exponent, then there is no sequence $(D_n)_{n < \omega}$ of definable sets such that:*

- (1) D_n is symmetric and $D_{n+1} + D_{n+1} \subseteq D_n$ for all $n < \omega$;
- (2) D_{n+1} is generic in D_n for all $n < \omega$;
- (3) $\bigcap_{n < \omega} D_n$ is not an intersection of definable groups.

Proof. Assume that there is such a sequence $(D_n)_{n < \omega}$ of definable sets. Replacing D_n by D_{n+1} if necessary, we can assume that D_0 is an approximate subgroup (i.e. finitely many translates of D_0 cover $D_0 + D_0$), because D_1 is an approximate subgroup by (1) and (2). We denote $D_0^{+n} := D_0 + \dots + D_0$. Then, $\langle D_0 \rangle = \bigcup_{n < \omega} D_0^{+n}$ is a \bigvee -definable group and, by (1), (2), and the assumption that D_0 is an approximate subgroup, we see that $\bigcap_{k < \omega} D_k \leq \langle D_0 \rangle$ is a \bigwedge -definable subgroup of bounded index. Hence,

$$H := \langle D_0 \rangle / \bigcap_{k < \omega} D_k$$

is a locally compact group with the logic topology (in which closed sets are defined as those whose preimages under the quotient map have type-definable intersections with all sets D_0^{+n} , $n < \omega$; see [HPP08, Lemma 7.5]). Since H is a torsion group, it follows from [Arm81, Theorem 3.5] that H has a basis $(H_i)_{i \in I}$ of neighbourhoods of the identity consisting of clopen subgroups. Since each $D_n / \bigcap_{k < \omega} D_k$ is a neighborhood of the identity, there is $H_n \subseteq D_n / \bigcap_{k < \omega} D_k$ which is a clopen subgroup

of H . Let $\pi : \langle D_0 \rangle \rightarrow H$ be the quotient map. Then,

$$\pi^{-1}[H_n] \subseteq D_n + \bigcap_{k < \omega} D_k \subseteq D_n + D_n \subseteq D_{n-1}$$

is a \bigwedge -definable group. Since $\pi^{-1}[H_n]^c \cap D_{n-1}$ is also \bigwedge -definable, we deduce that $\pi^{-1}[H_n]$ is a definable group laying between $\bigcap_{k < \omega} D_k$ and D_{n-1} . Since this is true for any $n > 0$, we get a contradiction with (3). \square

The following corollary yields some hints on how an example of finite exponent could be constructed.

Corollary 5.9. *If G is abelian of finite exponent, then the condition $G^{st,0} \neq G^{st} \neq G^{00}$ is equivalent to the existence of a sequence $(D_n)_{n < \omega}$ of definable sets such that:*

- (1) D_n is symmetric and $D_{n+1} + D_{n+1} \subseteq D_n$, for all $n < \omega$;
- (2) D_{n+1} is not generic in D_n for all $n < \omega$;
- (3) $\bigcap_{n < \omega} D_n$ is not an intersection of definable groups;
- (4) $[G : \bigcap_{n < \omega} D_n]$ is unbounded;
- (5) $G / \bigcap_{n < \omega} D_n$ is stable.

Proof. From Proposition 5.6, we obtain that the condition $G^{st,0} \neq G^{st} \neq G^{00}$ is equivalent to the existence of a sequence $(D_n)_{n < \omega}$ satisfying (1), (3), (4), and (5). Furthermore, by the previous proposition, such a sequence $(D_n)_{n < \omega}$ must contain an (infinite) subsequence satisfying (2). \square


Remark 5.10. *This section could be naturally generalized to the context of a \bigwedge -definable group G . This would require checking a few things, mainly that Hrushovski's theorem (i.e. [Pil96, Ch. 1, Lemma 6.18]) is valid for a stable \bigwedge -definable group (not necessarily living in a stable theory). We leave it to the reader.*


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INSTYTUT MATEMATYCZNY UNIwersytetu Wrocławskiego, PL. GRUNWALDZKI 2, 50-384 WROCLAW, POLAND

KRZYSZTOF KRUPIŃSKI  <https://orcid.org/0000-0002-2243-4411>
 Email address: Krzysztof.Krupinski@math.uni.wroc.pl

ADRIÁN PORTILLO  <https://orcid.org/0000-0001-9354-8574>
 Email address: Adrian.Portillo-Fernandez@math.uni.wroc.pl