Stable groups, Problem 4.1

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In this note we show quantifier elimination of the following theory. Let G be an infinite abelian group of exponent 2 (infinite vector space over $\mathbb{Z}/2\mathbb{Z}$), $A_1 \subseteq G$ an infinite linearly independent subset, A_n the set of all sums of distinct n elements from A_1, and $A_0 = \{0\}$. We consider structure $(G, +, A_n)_{n<\omega}$ with theory T and we work in an $\aleph_0$-saturated model M of T. (Actually, everything in this note holds for any model M, we only need $\aleph_0$-saturation to conclude quantifier elimination at the end.) M is an infinite vector space over $\mathbb{Z}/2\mathbb{Z}$ too, A_1^M is infinite linearly independent subset, A_n^M is the set of all sums of n distinct elements from A_1^M. Further on, A_n denotes A_n^M.

0.1. **Fact.** $\mathrm{span}(A_1) = \bigcup_{n<\omega} A_n$.

0.2. **Definition.** If $a \in \mathrm{span}(A_1)$, $S(a) = \{a_1, \dots, a_n\}$ where $a = a_1 + \cdots + a_n$ for $a_1, \dots, a_n \in A_1$. (So, $|S(a)| = n$ iff $a \in A_n$.)

In the following claims we will manipulate with sets $S(a)$ and their complements, so let us emphasize that $S(a)^c$ denotes the complement of $S(a)$ in A_1: $S(a)^c = A_1 \setminus S(a)$.

0.3. **Claim.** For $a, b \in \mathrm{span}(A_1)$, $S(a+b) = (S(a) \cap S(b)^c) \cup (S(a)^c \cap S(b))$.

Proof. Let $S(a) \cap S(b) = \{c_1, \dots, c_k\}$, $S(a) = \{c_1, \dots, c_k, a_1, \dots, a_m\}$ and $S(b) = \{c_1, \dots, c_k, b_1, \dots, b_n\}$, where c_i's, a_i's and b_i's are in A_1. Then $S(a+b) = \{a_1, \dots, a_m, b_1, \dots, b_n\}$, so the conclusion follows. □

0.4. **Definition.** A tuple $\bar{e} \in 2^n$ is *odd* if odd many coordinates are 1, and *even* otherwise.

0.5. **Claim.** For all $a_1, \dots, a_n \in \mathrm{span}(A_1)$:

$$S\left(\sum_{i=1}^n a_i\right) = \bigcup_{\substack{\bar{e} \in 2^n \\ \text{odd}}} \bigcap_{i=1}^n S(a_i)^{e_i} \quad \text{and} \quad S\left(\sum_{i=1}^n a_i\right)^c = \bigcup_{\substack{\bar{e} \in 2^n \\ \text{even}}} \bigcap_{i=1}^n S(a_i)^{e_i}.$$

Proof. Induction on n. For $n = 1$ the claim is obvious. Assume that the claim holds for n and take $a_1, \dots, a_n, a_{n+1}$. By Claim 0.3 we have:

$$\begin{aligned} S\left(\sum_{i=1}^{n+1} a_i\right) &= \left(S\left(\sum_{i=1}^n a_i\right) \cap S(a_{n+1})^c\right) \cup \left(S\left(\sum_{i=1}^n a_i\right)^c \cap S(a_{n+1})\right) \\ &= \left(\bigcup_{\substack{\bar{e} \in 2^n \\ \text{odd}}} \bigcap_{i=1}^n S(a_i)^{e_i} \cap S(a_{n+1})^c\right) \cup \left(\bigcup_{\substack{\bar{e} \in 2^n \\ \text{even}}} \bigcap_{i=1}^n S(a_i)^{e_i} \cap S(a_{n+1})\right) \\ &= \bigcup_{\substack{\bar{e} \in 2^{n+1} \\ \text{odd}}} \bigcap_{i=1}^{n+1} S(a_i)^{e_i}, \end{aligned}$$

so $S\left(\sum_{i=1}^{n+1} a_i\right)^c = \bigcup_{\substack{\bar{e} \in 2^{n+1} \\ \text{even}}} \bigcap_{i=1}^{n+1} S(a_i)^{e_i}$ as well. □

0.6. **Claim.** If $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$ in $\operatorname{span}(A_1)$ are such that $\bar{a} \equiv_{qf} \bar{b}$ then for every $\bar{e} \in 2^n$:

$$\left| \bigcap_{i=1}^{n} S(a_i)^{e_i} \right| = \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right|.$$

Proof. Induction on n. For $n = 1$, $a_1 \equiv_{qf} b_1$ implies $|S(a_1)| = |S(b_1)|$ as $a_1 \in A_{|S(a_1)|}$ and $b_1 \in A_{|S(b_1)|}$. Therefore, $|S(a_1)^c| = |S(b_1)^c|$ holds as well. Assume that $n > 1$. Let k be such that:

$$\left| \bigcap_{i=1}^{n} S(a_i) \right| = \left| \bigcap_{i=1}^{n} S(b_i) \right| + k.$$

(This is the intersection corresponding to $\bar{e} = (1, \ldots, 1)$.) Denote by $\delta(\bar{e})$ the number of 1's in $\bar{e}$. By induction on $n - \delta(\bar{e})$ we prove that:

$$\left| \bigcap_{i=1}^{n} S(a_i)^{e_i} \right| = \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right| + (-1)^{n-\delta(\bar{e})} k.$$

The assertion is true for $\delta(\bar{e}) = n$ by the definition of k. Consider $\bar{e}$ with $\delta(\bar{e}) < n$. Take one j such that $e_j = 0$, and denote by $\bar{e}'$ the tuple $\bar{e}$ with j-th coordinate swapped by 1, so $\delta(\bar{e}') = \delta(\bar{e}) + 1$. We have:

$$\left| \bigcap_{i=1}^{n} S(a_i)^{e_i} \right| + \left| \bigcap_{i=1}^{n} S(a_i)^{e'_i} \right| = \left| \bigcap_{\substack{i=1 \\ i \neq j}}^{n} S(a_i)^{e_i} \right| \overset{(*)}{=} \left| \bigcap_{\substack{i=1 \\ i \neq j}}^{n} S(b_i)^{e_i} \right| = \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right| + \left| \bigcap_{i=1}^{n} S(b_i)^{e'_i} \right|,$$

where $(*)$ holds by the first induction hypothesis. By the second induction hypothesis we have:

$$\left| \bigcap_{i=1}^{n} S(a_i)^{e'_i} \right| = \left| \bigcap_{i=1}^{n} S(b_i)^{e'_i} \right| + (-1)^{n-\delta(\bar{e}')} k,$$

hence we get:

$$\left| \bigcap_{i=1}^{n} S(a_i)^{e_i} \right| = \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right| + (-1)^{n-1-\delta(\bar{e}')} k = \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right| + (-1)^{n-\delta(\bar{e})} k.$$

This finishes the second induction.

By $\bar{a} \equiv_{qf} \bar{b}$ we have $\sum_{i=1}^{n} a_i \equiv_{qf} \sum_{i=1}^{n} b_i$, in particular $|S(\sum_{i=1}^{n} a_i)| = |S(\sum_{i=1}^{n} b_i)|$ (by the induction basis). On the other hand by Claim 0.5:

$$\left| S\left(\sum_{i=1}^{n} a_i \right) \right| = \sum_{\substack{\bar{e} \in 2^n \\ odd}} \left| \bigcap_{i=1}^{n} S(a_i)^{e_i} \right| = \sum_{\substack{\bar{e} \in 2^n \\ odd}} \left| \bigcap_{i=1}^{n} S(b_i)^{e_i} \right| + (-1)^{n-\delta(\bar{e})} k =$$

$$= \left| S\left(\sum_{i=1}^{n} b_i \right) \right| + \sum_{\substack{\bar{e} \in 2^n \\ odd}} (-1)^{n-\delta(\bar{e})} k,$$

so we conclude $k = 0$ as for odd $\bar{e}$, $(-1)^{n-\delta(\bar{e})}$ has the constant value. This finishes the proof. □

0.7. **Claim.** T has quantifier elimination.

Proof. It is enough for $\bar{a} \equiv_{qf} \bar{b}$ to find an automorphism $f \in \operatorname{Aut}(M)$ such that $f(\bar{a}) = \bar{b}$; fix such $\bar{a}$ and $\bar{b}$. Let $\bar{a}_1$ be a basis of $\operatorname{span}(A_1) \cap \operatorname{span}(\bar{a})$ and choose $\bar{a}_2$ such that $\bar{a}_1\bar{a}_2$ is a basis for $\operatorname{span}(\bar{a})$. Then $A_1\bar{a}_2$ is linearly independent as otherwise some linear combination of $\bar{a}_2$ belongs to $\operatorname{span}(A_1)$ but also to $\operatorname{span}(\bar{a})$, so to $\operatorname{span}(\bar{a}_1)$ which is not possible.

Since $\operatorname{span}(\bar{a}) = \operatorname{span}(\bar{a}_1\bar{a}_2)$ we see that $\bar{a}_1\bar{a}_2 = \varphi(\bar{a})$ and $\bar{a} = \psi(\bar{a}_1\bar{a}_2)$, where φ and ψ are coordinatewise linear combinations. Denote $\bar{b}_1\bar{b}_2 = \varphi(\bar{b})$; since „$\bar{x} = \psi(\varphi(\bar{x}))$" belongs to $\operatorname{tp}_{\mathrm{qf}}(\bar{a})$, we have

$\bar{b} = \psi(\bar{b}_1\bar{b}_2)$. Similarly, for $\theta(\bar{x}, \bar{x}_1, \bar{x}_2) \in \mathrm{tp}_{\mathrm{qf}}(\bar{a}\bar{a}_1\bar{a}_2)$ we have „$\theta(\bar{x}, \varphi(\bar{x}))$" belongs to $\mathrm{tp}_{\mathrm{qf}}(\bar{a})$, so we obtain $\theta(\bar{x}, \bar{x}_1, \bar{x}_2) \in \mathrm{tp}_{\mathrm{qf}}(\bar{b}, \bar{b}_1, \bar{b}_2)$, and $\bar{a}\bar{a}_1\bar{a}_2 \equiv_{qf} \bar{b}\bar{b}_1\bar{b}_2$ follows. In particular $\bar{b}_1\bar{b}_2$ are linearly independent and $\bar{b}_1 \in \mathrm{span}(A_1)$. Furthermore, $\mathrm{span}(\bar{b}) = \mathrm{span}(\bar{b}_1\bar{b}_2)$ as $\mathrm{span}(\bar{a}) = \mathrm{span}(\bar{a}_1\bar{a}_2)$ is expressible as a quantifier-free sentence over $\bar{a}\bar{a}_1\bar{a}_2$. Moreover, $\mathrm{span}(A_1) \cap \mathrm{span}(\bar{b}) = \mathrm{span}(\bar{b}_1)$: ($\supseteq$) is clear; for ($\subseteq$) if some linear combination $t(\bar{b})$ belongs to $\mathrm{span}(A_1)$, say to A_n, then „$t(\bar{x}) \in A_n$" is in $\mathrm{tp}_{\mathrm{qf}}(\bar{b})$ so $t(\bar{a}) \in \mathrm{span}(A_1)$ hence $t(\bar{a}) = s(\bar{a}_1)$ for some linear combination $s(\bar{a}_1)$. Formula „$t(\bar{x}) = s(\bar{x}_1)$" is in $\mathrm{tp}_{\mathrm{qf}}(\bar{a}\bar{a}_1)$ so $t(\bar{b}) = s(\bar{b}_1) \in \mathrm{span}(\bar{b}_1)$. Therefore, $A_1\bar{b}_2$ is linearly independent by the same reason as above.

Let $\bar{a}_1 = (a_{11}, \ldots, a_{1n})$ and $\bar{b}_1 = (b_{11}, \ldots, b_{1n})$. By Claim 0.6 we can find $f \in \mathrm{Sym}(A_1)$ such that f maps $\bigcap_{i=1}^n S(a_{1i})^{e_i}$ to $\bigcap_{i=1}^n S(b_{1i})^{e_i}$ for every $\bar{e} \in 2^n$. Then f can be extended to an automorphism of vector space $\mathrm{span}(A_1)$. Since a_{1j} is the sum of elements in sets $\bigcap_{i=1}^n S(a_{1i})^{e_i}$ for $\bar{e} \in 2^n$ with $e_j = 1$, $f(a_{1j})$ is equal to the sum of elements in sets $\bigcap_{i=1}^n S(b_{1i})^{e_i}$ for $\bar{e} \in 2^n$ with $e_j = 1$, i.e. $f(a_{1j}) = b_{1j}$; hence $f(\bar{a}_1) = \bar{b}_1$. Moreover, f preserves each A_n. Since $\bar{a}_2$ and $\bar{b}_2$ are independent over $\mathrm{span}(A_1)$, f can be further extended to an automorphism of vector space M such that $f(\bar{a}_2) = \bar{b}_2$. Clearly, $f \in \mathrm{Aut}(M)$. Since, $\bar{a} = \psi(\bar{a}_1\bar{a}_2)$ and $\bar{b} = \psi(\bar{b}_1\bar{b}_2)$ we get $f(\bar{a}) = \bar{b}$. □

0.8. **Corollary.** For any model M (or just vector subspace M) and $p \in S_1(M)$:

$$\{x \in a + A_n, x \notin a + A_n \mid n < \omega, a \in M\} \cap p(x) \vdash p(x).$$

Proof. Since $x = a$ is equivalent to $x \in a + A_0$ and M is a model, every atomic formula over M is given by $x \in a + A_n$ for $n < \omega$ and $a \in M$. Conclusion follows by quantifier elimination. □

We aim to describe complete 1-types over a model M. Fix M and a monster $\mathfrak{C} \succ M$.

0.9. **Claim.** There is a unique type $p \in S_1(M)$ containing $x \notin a + A_n$ for every $n < \omega$ and $a \in M$.

Proof. First note that for $n < \omega$ and $a \in M$ either $(a + A_n^M) \cap \mathrm{span}(A_1^M) = \emptyset$ or there is $m < \omega$ such that $a + A_n^M \subseteq \bigcup_{i<m} A_i^M$. If $a \notin \mathrm{span}(A_1^M)$ then clearly $(a + A_n^M) \cap \mathrm{span}(A_1^M) = \emptyset$. If $a \in \mathrm{span}(A_1^M)$, then $a \in A_k^M$ for some $k < \omega$, so $a + A_n^M \subseteq A_k^M + A_n^M \subseteq \bigcup_{i \leqslant k+n} A_i^M$.

Let us notice that $\{x \notin a + A_n \mid n < \omega, a \in M\}$ is consistent. For $n_1, \ldots, n_k < \omega$ and $a_1, \ldots, a_k \in M$ take $m < \omega$ such that either $a_i + A_{n_i}^M \subseteq \bigcup_{j<m} A_j^M$ or $(a_i + A_{n_i}^M) \cap \mathrm{span}(A_1) = \emptyset$ for every $i \leqslant k$. Then any element from A_m^M satisfies $x \notin a_i + A_{n_i}$ for $i \leqslant k$. Therefore, $\{x \notin a + A_n \mid n < \omega, a \in M\}$ is finitely consistent, hence consistent.

The type p is uniquely determined by Corollary 0.8. □

0.10. **Claim.** Let $q \in S_1(M)$, $q \neq p$. Denote by n_q the minimal $n < \omega$ such that $x \in a + A_n$ is in q for some $a \in M$.

(1) If $g \models q$ in $\mathfrak{C}$ and $x \in a + A_{n_q}$ is in q, then $g = a + c_1 + \ldots + c_{n_q}$ for some distinct $c_1, \ldots, c_{n_q} \in A_1^{\mathfrak{C}} \setminus A_1^M$.
(2) The element $a \in M$ such that $x \in a + A_{n_q}$ is in q is uniquely determined; we denote it by a_q.
(3) The pair (n_q, a_q) determines q.
(4) For any distinct $c_1, \ldots, c_{n_q} \in A_1^{\mathfrak{C}} \setminus A_1^M$, $a_q + c_1 + \cdots + c_{n_q} \models q$.

Proof. (1) We can write $g = a + c_1 + \cdots + c_{n_q}$ for some $c_1, \ldots, c_{n_q} \in A_1^{\mathfrak{C}}$. If $c_1 \in M$, then $g = a' + c_2 + \cdots + c_{n_q} \in a' + A_{n_q-1}^{\mathfrak{C}}$ where $a' = a + c_1 \in M$, so $x \in a' + A_{n_q-1}$ is in q which contradicts the minimality of n_q. Thus $c_1 \notin M$. Similarly, all $c_1, \ldots, c_{n_q} \notin M$.

(2) Let $x \in a + A_{n_q}, x \in b + A_{n_q}$ be in q and $g \models q$ in $\mathfrak{C}$. By (1) we can write $g = a + c_1 + \cdots + c_{n_q} = b + d_1 + \cdots + d_{n_q}$ for some distinct $c_1, \ldots, c_{n_q} \in A_1^{\mathfrak{C}} \setminus A_1^M$ and distinct $d_1, \ldots, d_{n_q} \in A_1^{\mathfrak{C}} \setminus A_1^M$. Then $a + b = c_1 + \cdots + c_{n_q} + d_1 + \cdots + d_{n_q}$ belongs to M, which is possible only if $\{c_1, \ldots, c_{n_q}\} = \{d_1, \ldots, d_{n_q}\}$, i.e. $a + b = 0$. Thus $a = b$.

(3) Let $r \in S_1(M)$ be such that $r \neq p$ and $(n_r, a_r) = (n_q, a_q) =: (n, a)$. Let $g \models q$, $h \models r$. By (1) we can write $g = a + c_1 + \cdots + c_n$ and $h = a + d_1 + \cdots + d_n$ for distinct $c_1, \ldots, c_n \in A_1^{\mathfrak{C}} \setminus A_1^M$ and distinct $d_1, \ldots, d_n \in A_1^{\mathfrak{C}} \setminus A_1^M$. Note that c_i's and d_i's, as well as their linear combinations are not in M. Thus $\mathrm{tp}_{\mathrm{qf}}(\bar{c}/M) = \mathrm{tp}_{\mathrm{qf}}(\bar{d}/M)$. By quantifier elimination $\mathrm{tp}(\bar{c}/M) = \mathrm{tp}(\bar{d}/M)$. So id_M can be extended to $f \in \mathrm{Aut}(\mathfrak{C})$ such that $f(c_i) = d_i$. Then $f(g) = h$ and hence $r = q$.

(4) By (1) and (the proof of) (3). □

0.11. **Corollary.** T is ω-stable.

Proof. By Claim 0.9 and Claim 0.10 for a countable model M, $S_1(M)$ is countable. This is enough. □

0.12. **Corollary.** Let $q \in S_1(M)$ be such that $q \neq p$ and $(n_q, a_q) = (n, 0)$. Then $x \in a + A_m$ belongs to q iff $n \leqslant m$ and $a \in A_{m-n}^M$.

Proof. By Claim 0.10(1), there are distinct $c_1, \ldots, c_n \in A_1^{\mathfrak{C}} \setminus A_1^M$ such that $c_1 + \cdots + c_n \models q$. Assume that $x \in a + A_m$ is in q. By the definition of n, $n \leqslant m$. Now $c_1 + \cdots + c_n \in a + A_m^{\mathfrak{C}}$ so we can write $a = c_1 + \cdots + c_n + d_1 + \cdots + d_m$ where $d_1, \ldots, d_m \in A_1^{\mathfrak{C}}$ are distinct. Since this sum is in M, the only possibility is that $\{c_1, \ldots, c_n\} \subseteq \{d_1, \ldots, d_m\}$ and $\{d_1, \ldots, d_m\} \setminus \{c_1, \ldots, c_n\} \subseteq A_1^M$. Therefore $a \in A_{m-n}^M$. On the other hand, if $a \in A_{m-n}^M$ then $a + c_1 + \cdots + c_n \in A_m^{\mathfrak{C}}$, so $c_1 + \cdots + c_n$ satisfies $x \in a + A_m$. □

Further on we will write $q_{(n,a)}$ for a type $q \in S_1(M)$ such that $q \neq p$ and $(n_q, a_q) = (n, a)$.

0.13. **Claim.** We work in $\mathfrak{C}$.

(1) $\mathrm{RM}(A_{n+1}) > \mathrm{RM}(A_n)$ and $\mathrm{RM}(A_{n+1}) \geqslant n + 1$ for $n < \omega$;
(2) in fact, $\mathrm{RM}(A_n) = n$ for $n < \omega$ and $\mathrm{RM}(q_{(n,a)}) = n$ for $n < \omega, a \in \mathfrak{C}$;
(3) $\mathrm{RM}(A_n^c) = \omega$ for $n < \omega$ and $\mathrm{RM}(p) = \omega$;
(4) $\mathrm{RM}(x = x) = \omega$.

Proof. (1) We proceed by induction on n. For $n = 0$, the assertion is trivial as A_0 is finite and A_1 is infinite. Let $n \geqslant 1$. Note that by ω-stability all RM's are ordinal. For $a \in A_1$ denote by $A_n(a)$ the subset of A_n consisting of all sums of n-distinct elements from A_1 which include a, and by $B_n(a)$ the complement $A_n \setminus A_n(a)$. Note that $a + A_n = (a + A_n(a)) \cup (a + B_n(a))$, $a + A_n(a) \subseteq A_{n-1}$ and $a + B_n(a) \subseteq A_{n+1}$; by induction hypothesis $\mathrm{RM}(a + A_n(a)) \leqslant \mathrm{RM}(A_{n-1}) < \mathrm{RM}(A_n)$, so $\mathrm{RM}(a + B_n(a)) = \mathrm{RM}(A_n)$. Also for distinct $a, b \in A_1$, $(a + B_n(a)) \cap (b + B_n(b)) \subseteq a + b + A_{n-1}$, so by induction hypothesis again $\mathrm{RM}((a + B_n(a)) \cap (b + B_n(b))) < \mathrm{RM}(A_n)$. Take distinct $a_i \in A_1$, $i < \omega$ and consider:

$$S_i = (a_i + B_n(a_i)) \setminus \bigcup_{j<i} (a_j + B_n(a_j)).$$

S_i's are clearly mutually disjoint subsets of A_{n+1}. Moreover, $\mathrm{RM}(S_i) = \mathrm{RM}(A_n)$ since it is obtained by excluding a finite union of sets of $\mathrm{RM} < \mathrm{RM}(A_n)$ from a set of $\mathrm{RM} = \mathrm{RM}(A_n)$. Therefore, $\mathrm{RM}(A_{n+1}) \geqslant \mathrm{RM}(A_n) + 1 > \mathrm{RM}(A_n)$.

The second assertion now obviously holds by the induction hypothesis.

(2) We show by induction that $\mathrm{RM}(A_n) = n$ and $\mathrm{RM}(q_{(n,a)}) = n$. For $n = 0$ this is clear. Let $n > 0$. Note that each type in $[A_n] \subseteq S_1(\mathfrak{C})$ is of the form $q_{(m,a)}$ for some $m \leqslant n$ and $a \in \mathfrak{C}$. By induction hypothesis, for $m < n$ we have $\mathrm{RM}(q_{(m,a)}) = m < n$. On the other hand, for $m = n$ the element a must be equal to 0 by Claim 0.10(2) (as $x \in A_n$ and $x \in a + A_n$ are both in $q_{(m,a)}$), so in $[A_n]$ there is at most only one type whose RM is not less than n. Hence, $\mathrm{RM}(A_n) \leqslant n$.

Thus, by (1), $\mathrm{RM}(A_n) = n$. Since $[A_n]$ contains a type with $\mathrm{RM} = \mathrm{RM}(A_n)$, by the previous paragraph we conclude $\mathrm{RM}(q_{(n,0)}) = n$. By Claim 0.10 we may conclude $q_{(n,a)} = a + q_{(n,0)}$, so $\mathrm{RM}(q_{(n,a)}) = \mathrm{RM}(q_{(n,0)}) = n$.

(3) Since A_n^c contains A_m for $m > n$, we have $\mathrm{RM}(A_n^c) \geqslant \mathrm{RM}(A_m) = m$ for $m > n$, hence $\mathrm{RM}(A_n^c) \geqslant \omega$. As almost all types in $[A_n^c] \subseteq S_1(\mathfrak{C})$, except for maybe p, are of finite RM by (2), we have $\mathrm{RM}(A_n^c) \leqslant \omega$. Thus $\mathrm{RM}(A_n^c) = \omega$. Consequently, $\mathrm{RM}(p) = \omega$ as p is the only candidate for $\mathrm{RM} = \mathrm{RM}(A_n^c)$ in $[A_n^c]$.

(4) Clear. □

0.14. **Corollary.** If $(n_i)_{i<\omega}$ is an increasing sequence of positive integers, then $\lim q_{(n_i,0)} = p$ in $S_1(M)$.

Proof. Let $r \in S_1(M)$ be an accumulation point of the sequence $(q_{(n_i,0)})_{i<\omega}$. If $\phi(x) \in L(M)$ is a formula of a finite RM, then $[\phi(x)]$ contains only finitely many members of the sequence as their ranks n_i's increase. Thus $\phi(x) \notin r$. Therefore $r = p$. □

0.15. **Claim.** Let $M \prec \mathfrak{C}$. Then $p(\mathfrak{C})$ generates $\mathfrak{C}$, where we consider $p \in S_1(M)$.

Proof. Let $g \models p$. First we claim that $M \subseteq \mathrm{span}(p(\mathfrak{C}))$. Let $m \in M$ and consider $\mathrm{tp}(m + g/M)$. If it is p, then $m = g + (m + g) \in \mathrm{span}(p(\mathfrak{C}))$. Otherwise $\mathrm{tp}(m + g/M) = q_{(n,a)}$ for some $n < \omega$ and $a \in M$, so by Claim 0.10(1) we can write $m + g = a + c_1 + \cdots + c_n$ for distinct $c_1, \ldots, c_n \in A_1^{\mathfrak{C}} \setminus A_1^M$, hence $g = m + a + c_1 + \cdots + c_n$ satisfies $x \in m + a + A_n$; a contradiction. Further we claim $A_1^{\mathfrak{C}} \subseteq \mathrm{span}(p(\mathfrak{C}))$. Let $c \in A_1^{\mathfrak{C}}$ and consider $\mathrm{tp}(c + g/M)$. If it is p, then $c = g + (c + g) \in \mathrm{span}(p(\mathfrak{C}))$. Otherwise $\mathrm{tp}(c + g/M) = q_{(n,a)}$ for some $n < \omega$ and $a \in M$, so as before we write $c + g = a + c_1 + \cdots + c_n$, hence $g = a + c + c_1 + \cdots + c_n$ satisfies either $x \in a + A_{n-1}$ (if c equals one of c_i's) or $x \in a + A_{n+1}$ (if c differs from all c_i's); in both cases we have a contradiction.

Finally, we prove that $p(\mathfrak{C})$ generates $\mathfrak{C}$. Let $h \in \mathfrak{C}$ and consider $\mathrm{tp}(h + g/M)$. If it is p, then $h = g + (h + g) \in \mathrm{span}(p(\mathfrak{C}))$. Otherwise, $\mathrm{tp}(h + g/M) = q_{(n,a)}$, so as above we write $h + g = a + c_1 + \cdots + c_n$. Then $h = a + c_1 + \cdots + c_n + g \in \mathrm{span}(p(\mathfrak{C}))$ by the previous paragraph. □

0.16. **Claim.** If $H \leqslant G$ is a proper definable subgroup, then H is finite.

Proof. Suppose that H is infinite and consider $[H]$ in $S_1(G)$; we claim that $p \in [H]$. Since H is infinite, there is a non-algebraic type $r \in [H]$. If $r = p$ we are done. Otherwise $r = q_{(n,a)}$ for some $n \geqslant 1$ and $a \in G$. Then $q_{(n,a)}(\mathfrak{C}) \subseteq H^{\mathfrak{C}}$. For distinct $c, d, c_2, \ldots, c_n \in A_1^{\mathfrak{C}} \setminus A_1^G$, by Claim 0.10 $a + c + c_2 + \cdots + c_n$ and $a + d + c_2 + \cdots + c_n$ satisfy $q_{(n,a)}$, so they are in $H^{\mathfrak{C}}$, hence their sum $c + d \in H^{\mathfrak{C}}$ too. Now, for distinct $c_1, c_2, \ldots, d_1, d_2, \ldots \in A_1^{\mathfrak{C}} \setminus A_1^G$ we have $c_1 + \cdots + c_k + d_1 + \cdots + d_k \in H^{\mathfrak{C}}$ for all $k < \omega$. Since $\mathrm{tp}(c_1 + \cdots + c_k + d_1 + \cdots + d_k/G) = q_{(2k,0)}$ by Claim 0.10, we conclude $q_{(2k,0)} \in [H]$ for all $k < \omega$. Since $[H]$ is closed by Corollary 0.14, $p \in [H]$.

Since $p \in [H]$, $p(\mathfrak{C}) \subseteq H^{\mathfrak{C}}$, so $H^{\mathfrak{C}} = \mathfrak{C}$ by Claim 0.15. Therefore $H = G$; a contradiction. □

0.17. **Comment.** The assumption $\mathrm{RM}(G) < \omega$ in Zilber's theorem is necessary. The set $A_0 \cup A_1$, which contains 0, is indecomposable since it is infinite, but every definable subgroup of G is either G or finite by Claim 0.16. On the other hand, $\mathrm{span}(A_0 \cup A_1)$ can't be generated in finitely many steps.