On relationships between algebraic properties of groups and rings in some model-theoretic contexts

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Abstract

We study relationships between certain algebraic properties of groups and rings definable in a first order structure or $*$-closed in a compact G-space. As a consequence, we obtain a few structural results about ω-categorical rings as well as about small, nm-stable compact G-rings, and we also obtain surprising relationships between some conjectures concerning small profinite groups.

0   Introduction

One of the main themes in model theory is to classify interesting algebraic structures satisfying some natural model-theoretic assumptions. The algebraic structures that we concentrate on are groups and rings. Our model-theoretic/topological contexts will be mainly ω-categorical structures and small Polish structures (in particular, small profinite structures). In all these contexts, there are many results and conjectures describing the structure of groups or rings satisfying some extra model-theoretic assumptions. However, it seems that relationships between results and conjectures about groups and rings have not been fully described. So, the main goal of this paper is to analyze such relationships.

In Section 2, we study such relationships in a very general context of groups and rings definable in an arbitrary first order structure or $*$-closed in an arbitrary compact G-space. In both these contexts, we transfer some properties of groups (e.g. virtual commutativity or solvability) to the corresponding properties of rings (e.g. virtual triviality or nilpotency). As a consequence, using certain results on groups, in Section 3 we get structural results on ω-categorical rings and on small, nm-stable compact G-rings. For example, we prove that each small, nm-stable compact G-ring is nilpotent-by-finite. We also obtain surprising relationships between some conjectures on small profinite groups. In our investigations, besides results proved in

Section 2, we will also need to use [9, Theorem 3.5] which allows one to deduce certain property of groups (virtual nilpotency of solvable small profinite groups) from the corresponding property of rings (virtual nilpotency of small profinite rings). Theorem 3.5 from [9] is restricted to the context of small profinite structures. At the end of the paper, we prove a counterpart of this result for ω-categorical structures.

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1 Preliminaries

Recall that a first order structure M in a countable language is said to be ω-categorical if, up to isomorphism, $Th(M)$ has at most one model of cardinality $\aleph_0$.

Now, we recall some notions concerning Polish structures. For more details see [5]. A Polish structure is a pair (X, G), where G is a Polish group acting faithfully on a set X so that the stabilizers of all singletons are closed subgroups of G. We say that (X, G) is small if for every $n \in \omega$, there are only countably many orbits on X^n under the action of G. In a small Polish structure (X, G), there is a ternary relation, $\overset{nm}{\underset{}{\smile}}$, on finite tuples (or subsets) of X which has some nice properties (e.g. symmetry, transitivity, the existence of independent extensions), and which allows us to define the so-called $\mathcal{NM}$-rank on orbits of finite tuples over finite subsets of X in analogy with the Lascar U-rank. A Polish structure is said to be nm-stable if the $\mathcal{NM}$-rank of every orbit is an ordinal. For $Y \subseteq X^n$, we define $Stab(Y) := \{g \in G : g[Y] = Y\}$. We say that Y is invariant [over a finite set A] if $Stab(Y) = G$ [respectively, $Stab(Y) \supseteq G_A$, where G_A is the pointwise stabilizer of A].

For us, compact spaces and topological groups are Hausdorff by definition. A [compact] G-space is a Polish structure (X, G), where G acts continuously on a [compact] space X. If (X, G) is a G-space, we say that $D \subseteq X^n$ is A-closed (for a finite $A \subseteq X$) if it is closed and invariant over A. We say that it is $*$-closed, if it is A-closed for some finite A. Assume (X, G) is a compact G-space. We define X^{teq} (topological imaginary extension) as the disjoint union of the spaces X^n/E with E ranging over all $\emptyset$-closed equivalence relations on X^n. The spaces X^n/E are called topological sorts of X^{teq}. Then, each topological sort X^n/E together with the group $G/G_{X^n/E}$ is a compact $G/G_{X^n/E}$-space. If E is A-closed for some finite set A, then replacing G by G_A, X^n/E can also be treated as a topological sort.

Now, we recall some facts about groups. A [compact] G-group [G-ring] is a Polish structure (H, G), where G acts continuously and by automorphisms on a [compact] topological group [ring] H. We should mention here that by [16, Proposition 5.1.2], for a topological ring R, the following conditions are equivalent: (i) R is compact, (ii) R is profinite, (iii) there is a basis of open neighborhoods of 0 consisting of open ideals.

Let (X, G) be a G-space. We say that a group H is a $*$-closed in X [or in X^{teq} if X is compact] if both H and the group operation on H are $*$-closed in X [X^{teq},

respectively]. $*$-closed rings are defined analogously.

We work in a compact G-space (X, G). Assume H is $*$-closed in X^{teq}; for simplicity, $\emptyset$-closed (then $(H, G/G_H)$ is a compact G/G_H-group). Let $a \in H$ and $A \subseteq X$ be finite. We say that the orbit $o(a/A)$ is nm-generic (or that a is nm-generic over A) if for all $b \in H$ with $a \underset{A}{\overset{nm}{\smile}} b$, one has that $b \cdot a \underset{A}{\overset{nm}{\smile}} A, b$. It turns out that under the assumption of smallness of (X, G), nm-generics satisfy all the basic properties that generics satisfy in groups with a simple theory, including existence. More precisely, [5, Proposition 5.5] tells us that $o(a/A)$ is nm-generic iff $o(a/A) \subseteq_{nm} H$ (i.e. $o(a/A)$ is a non-meager subset of H), and by smallness such an orbit exists.

Profinite spaces will be always inverse limits of countable systems. A profinite structure is a compact G-space (X, G), where X is a profinite space and G is a compact (equivalently profinite) group. A profinite group [ring] regarded as profinite structure is a compact G-group [G-ring] (H, G), where H is a profinite group [ring] and G is a profinite group. Originally profinite structures were defined as pairs (X, G), where H is a profinite space and G is a closed subgroup of the group of all homeomorphisms of X preserving a distinguished inverse system defining X; similarly for profinite groups and rings. By [7, Proposition 1.5] together with [15, Proposition 1.4] and [9, Remark 2.9], both versions of the definitions are equivalent. This means that a profinite group [ring] regarded as profinite structure has a basis of open neighborhoods of e [0, respectively] consisting of clopen normal subgroups [ideals] invariant under G.

It was noticed in [5] that if (H, G) is a small compact G-group, then H is locally finite and hence profinite. However, G is only Polish (not necessarily compact), which makes the class of small compact G-groups much wider than the class of small profinite groups.

Recall some basic notions from ring theory. In this paper, all rings are associative, but they are not assumed to contain 1 or to be commutative. An element r of a ring R is nilpotent of nilexponent n if $r^n = 0$ and n is the smallest number with this property. The ring is nil [of nilexponent n] if every element is nilpotent [of nilexponent $\leq n$ and there is an element of nilexponent n]. The ring is nilpotent of class n if $r_1 \cdots r_n = 0$ for all $r_1, \ldots, r_n \in R$ and n is the smallest number with this property. An element r is null if $rR = Rr = \{0\}$. The ring is null if all its elements are. If $S \subseteq R$, then $Ann_R(S) = \{r \in R : rS = Sr = \{0\}\}$ is the two-sided annihilator of S in R. Note that $Ann_R(S)$ is always a subgroup of R^+, and if S is an ideal of R, then so is $Ann_R(S)$. $Ann_R^{left}(S)$ and $Ann_R^{right}(S)$ will denote the left and the right annihilator of S in R. By R^n we will denote the subring of R generated by all products $r_1 \ldots r_n$ for $r_1, \ldots, r_n \in R$. Notice that R^n is always an ideal. For groups [rings] G and H, $H \leq G$ means that H is a subgroup [subring] of G.

Recall that an abstract group is said to be abelian-by-finite [nilpotent-by-finite, solvable-by-finite] if it has a (normal) abelian [nilpotent, solvable] subgroup of finite index. Similarly, a ring is null-by-finite [nilpotent-by-finite] if it has a null [nilpotent] ideal of finite index. [12, Lemma 1] says that if S is a finite index subring of a ring R, then R has a finite index ideal contained in S. Thus, we have

Remark 1.1 *A ring is null-by-finite [nilpotent-by-finite] iff it has a null [nilpotent] subring of finite index.*

We say that a group G is (finite central)-by-abelian-by-finite if there are $H \trianglelefteq G$ and $K \trianglelefteq H$ such that G/H is finite, H/K is abelian, and K is a finite subgroup of $Z(H)$. A ring R is (finite null)-by-null-by-finite if there are $I \trianglelefteq R$ and $J \trianglelefteq I$ such that R/I is finite, I/J is null, and J is a finite subring of $Ann_I(I)$. Taking an intersection of appropriate centralizers, we easily see that G is (finite central)-by-abelian-by-finite iff it is finite-by-abelian-by-finite; a similar remark holds for rings as well.

There are many results describing the structure of ω-categorical groups and small compact G-groups. Here, we recall only a few of them, which will be useful in this paper (for a quick overview of some older results on ω-categorical groups and rings see [8]). The first one is [13, Theorem 1.2], and the second one is [5, Theorem 5.19].

Fact 1.2 *An ω-categorical group satisfying NSOP (the negation of the strict order property) is nilpotent-by-finite.*

Fact 1.3 *A small, nm-stable compact G-group is solvable-by-finite.*

In fact, using the above result, it was shown in [10] that small, nm-stable compact G-groups are nilpotent-by-finite. It is conjectured that they are even abelian-by-finite. A partial result in this direction is [10, Theorem 2.9]:

Fact 1.4 *Let (H,G) be a small compact G-group. If $\mathcal{NM}(H) < \omega$ or $\mathcal{NM}(H) = \omega^\alpha$ for some ordinal α, then H is abelian-by-finite.*

2 General context

In this section, $\mathcal{M}$ is a first order structure. First, we show that if all groups definable in $\mathcal{M}$ have certain algebraic properties, then all rings definable in $\mathcal{M}$ have some corresponding properties. Then, we notice that the same is true for groups and rings $*$-closed in a fixed compact G-space (X, G). Finally, after making a few remarks concerning definability of certain subgroups and ideals, we discuss reversed problems, i.e. whether certain properties of all definable rings imply the corresponding properties for all definable groups.

Theorem 2.1 *(i) If every group definable in $\mathcal{M}$ is solvable-by-finite, then every ring with identity, or of finite characteristic, definable in $\mathcal{M}$ is nilpotent-by-finite.*
(ii) If every nilpotent group definable in $\mathcal{M}$ is abelian-by-finite, then every ring definable in $\mathcal{M}$ is null-by-finite.
(iii) If every nilpotent group definable in $\mathcal{M}$ is (finite central)-by-abelian-by-finite, then every ring definable in $\mathcal{M}$ is (finite null)-by-null-by-finite.

Proof. (i) Let R be a ring definable in $\mathcal{M}$ which contains 1 or which is of finite characteristic. Suppose R has a finite characteristic c. Put $R_1 = R \times \mathbb{Z}_c$ (where $\mathbb{Z}_c$ is the ring of integers modulo c), and define $+$ and $\cdot$ on R_1 by

$$(a, k) + (b, l) = (a + b, k +_c l) \text{ and } (a, k) \cdot (b, l) = (ab + l \times a + k \times b, k \cdot_c l),$$

where $+_c$ and $\cdot_c$ are addition and multiplication modulo c, and $l \times a := a + \cdots + a$ (l-many times). This turns R_1 into a ring with 1 which is definable in $\mathcal{M}$ so that R is a finite index ideal in R_1. Thus, it is enough to consider the case when R contains 1.

Let $H = GL_3(R)$ be the group of all 3×3 invertible matrices with entries from R. Then, H is definable in $\mathcal{M}$, and so it is solvable-by-finite. Let H_0 be a normal, solvable, finite index subgroup of H.

For distinct $i, j \in \{1, 2, 3\}$, define $t_{ij}(\alpha)$ as the element of H with 1's on the diagonal, α on the (i, j)-th position, and 0's elsewhere. We have the following well-known formulas for multiplication and for commutators:

$$t_{ij}(\alpha)t_{ij}(\beta) = t_{ij}(\alpha + \beta) \text{ and } [t_{ik}(\alpha), t_{kj}(\beta)] = t_{ij}(\alpha\beta), \qquad (*)$$

for pairwise distinct $i, j, k \in \{1, 2, 3\}$.

Define

$$I = \{\alpha \in R : t_{ij}(\alpha) \in H_0 \text{ for all distinct } i, j \in \{1, 2, 3\}\}.$$

Using $(*)$ and the fact that H_0 is normal in H, we get that I is an ideal of R. We claim that I is of finite index in R. If not, then by Ramsey's theorem, there are α_i, $i \in \omega$, such that for some distinct $i, j \in \{1, 2, 3\}$ and for every $n > m$, we have $t_{ij}(\alpha_n - \alpha_m) \notin H_0$. But, $t_{ij}(\alpha_n - \alpha_m) = t_{ij}(\alpha_n)t_{ij}(\alpha_m)^{-1}$, and so H_0 is of infinite index in H, a contradiction.

By the solvability of H_0, there is n such that $H_0^{(n)} = \{e\}$. Then, $(*)$ implies that for any $\alpha_1, \ldots, \alpha_{2^n} \in I$ and distinct $i, j \in \{1, 2, 3\}$, we have $t_{ij}(\alpha_1 \ldots \alpha_{2^n}) \in H_0^{(n)} = \{e\}$, and so $\alpha_1 \ldots \alpha_{2^n} = 0$.

Thus, we have proved that I is a nilpotent ideal of finite index in R.

(ii) Let $H = UT_3(R)$ be the group of upper triangular 3×3 matrices with 1's on the diagonal (even if R does not have an identity, we can use an external 1, i.e. 1 from a superring with identity, which always exists) and elements from R above the diagonal. It is standard that H is nilpotent. We see that H is definable in $\mathcal{M}$, and so it is abelian-by-finite. Let H_0 be a normal, abelian subgroup of H of finite index.

Define

$$A = \left\{ (\alpha, \beta) \in R \times R : \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \in H_0 \text{ for some } \gamma \in R \right\}.$$

It is easy to check that A is a finite index subgroup of $R^+ \times R^+$. Let A_1 denote the projection of $A \cap (R^+ \times \{0\})$ on the first coordinate, and A_2 the projection of

$A \cap (\{0\} \times R^+)$ on the second coordinate. Then, A_1 and A_2 are finite index subgroups of R^+, and $A_1 \times A_2 \leq A$. Put $A_0 = A_1 \cap A_2$. We see that A_0 is a finite index subgroup of R^+, and $A_0 \times A_0 \leq A$. We will be done if we show that A_0 is a null subring of R.

For any $\alpha, \beta, \gamma, \alpha', \beta', \gamma' \in R$ we have

$$
\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha' & \gamma' \\ 0 & 1 & \beta' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \alpha + \alpha' & \gamma + \gamma' + \alpha\beta' \\ 0 & 1 & \beta + \beta' \\ 0 & 0 & 1 \end{pmatrix}
$$

Consider any $\alpha, \beta' \in A_0$. By the definition of A and the fact that H_0 is abelian, there are $\gamma, \gamma' \in R$ such that

$$
\begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \gamma' \\ 0 & 1 & \beta' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \gamma' \\ 0 & 1 & \beta' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha & \gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

This implies that $\alpha\beta' = 0$. Thus, A_0 is a null subring.

(iii) We argue in a similar way as in (ii). Let $H = UT_4(R)$ be the group of upper triangular 4×4 matrices with 1's on the diagonal (as before, if $1 \notin R$, we can use an external 1) and elements from R above the diagonal. Then, H is nilpotent. We see that H is definable in $\mathcal{M}$, and so it has a finite index, normal subgroup H_0 such that $[H_0, H_0]$ is finite and contained in the center of H_0.

Define A as the collection of all $(\alpha_{12}, \alpha_{23}, \alpha_{34}) \in R \times R \times R$ such that

$$
\begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & 1 & \alpha_{23} & \alpha_{24} \\ 0 & 0 & 1 & \alpha_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_0
$$

for some $\alpha_{13}, \alpha_{14}, \alpha_{24} \in R$.

One can check that A is a finite index subgroup of $R^+ \times R^+ \times R^+$. Let A_1, A_2 and A_3 be the projections of $A \cap (R \times \{0\} \times \{0\})$, $A \cap (\{0\} \times R \times \{0\})$ and $A \cap (\{0\} \times \{0\} \times R)$ on the first, on the second and on the third coordinate, respectively. Let $A_0 = A_1 \cap A_2 \cap A_3$. Then, A_0 is a finite index subgroup of R^+. Let D be the subgroup of R^+ generated by A_0 and $A_0 \cdot A_0$. We will be done if we show that D is a subring of R^+ which is (finite null)-by-null. For this, it is enough to prove that the subgroup of R^+ generated by $A_0 \cdot A_0$ is finite and that $A_0 \cdot A_0 \cdot A_0 = \{0\}$.

Consider any $\alpha, \beta \in A_0$. Then, we can find

$$
M := \begin{pmatrix} 1 & \alpha & \alpha_{13} & \alpha_{14} \\ 0 & 1 & 0 & \alpha_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_0 \quad \text{and} \quad N := \begin{pmatrix} 1 & 0 & \beta_{13} & \beta_{14} \\ 0 & 1 & \beta & \beta_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_0.
$$

After some computations, we get

$$[M, N] = \begin{pmatrix} 1 & 0 & \alpha\beta & \alpha\beta_{24} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in [H_0, H_0].$$

Since $[H_0, H_0]$ is finite, we get only finitely many possibilities for $\alpha\beta$, and so $A_0 \cdot A_0$ is finite. In fact, as all products of matrices (and their inverses) obtained above are still in $[H_0, H_0]$, we get that the subgroup of R^+ generated by $A_0 \cdot A_0$ is finite.

Consider any $\gamma \in A_0$. We can find $x, y, z \in R$ such that

$$P := \begin{pmatrix} 1 & 0 & x & z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} \in H_0$$

We know that $[M, N] \in Z(H_0)$, so $P[M, N] = [M, N]P$. Computing these matrices, we get

$$\begin{pmatrix} 1 & 0 & \alpha\beta + x & \alpha\beta_{24} + z \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & x + \alpha\beta & z + \alpha\beta\gamma + \alpha\beta_{24} \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & \gamma \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

This implies that $\alpha\beta\gamma = 0$, and so $A_0 \cdot A_0 \cdot A_0 = \{0\}$. ∎

Question 2.2 *Suppose that each solvable group definable in $\mathcal{M}$ is nilpotent-by-finite. Does it imply that every ring (with identity) definable in $\mathcal{M}$ is nilpotent-by-finite?*

By similar arguments, we get the following result.

Theorem 2.3 *Let (X, G) be a compact G-space.*
(i) If every group $*$-closed in X [or in X^{teq}] is solvable-by-finite, then every ring with identity, or of finite characteristic, $*$-closed in X [in X^{teq}] is nilpotent-by-finite.
(ii) If every nilpotent group $*$-closed in X [or in X^{teq}] is abelian-by-finite, then every ring $*$-closed in X [in X^{teq}] is null-by-finite.

Since each abelian, small compact G-group has finite exponent (see the proof of [9, Proposition 2.3]), each small, compact G-ring has finite characteristic. Thus, assuming smallness in the above theorem, the extra assumption about the ring in point (i) is automatically satisfied (in particular, this applies to small profinite structures).

Corollary 2.4 *Let (X, G) be a small compact G-space.*
(i) If every group $*$-closed in X [or in X^{teq}] is solvable-by-finite, then every ring $*$-closed in X [in X^{teq}] is nilpotent-by-finite.
(ii) If every nilpotent group $*$-closed in X [or in X^{teq}] is abelian-by-finite, then every ring $*$-closed in X [in X^{teq}] is null-by-finite.

The following remark is rather standard.

Remark 2.5 *Let G be any group.*
(i) If G has an abelian [normal] subgroup H of finite index, then it has a definable, abelian [normal] subgroup of finite index which contains H.
(ii) If G has a nilpotent [normal] subgroup H of finite index, then it has a definable, nilpotent [normal] subgroup of finite index which contains H.

Proof. (i) The centralizer $C(H)$ contains H, so it is the intersection of finitely many centralizers of elements of H. Thus, $Z(C(H)) \geq H$ is a definable, abelian [normal] subgroup of finite index in G.
(ii) Repeat the proof of [6, Remark 3.3(ii)] using the fact that if $[G : C(N)] < \omega$, then $C(N)$ is definable (this allows one to eliminate the application of icc). Notice that when G is countable, ω-categorical and H is normal, the Fitting subgroup of G (i.e. the group generated by all normal, nilpotent subgroups) does the job. ∎

Assuming icc on centralizers in definable quotients of definable subgroups, we have a variant of the above remark for solvability [6, Remark 3.3(i)]. However, it is not clear to me what happens without icc.

Question 2.6 *Is it true that if G has a solvable subgroup of finite index, then it has a definable, solvable, normal subgroup of finite index?*

The answer is positive for ω-categorical groups: in a countable elementary substructure, the group generated by all solvable, normal subgroups of finite index does the job.
We have the following counterpart of Remark 2.5 for rings.

Remark 2.7 *Let R be any ring.*
(i) If R has a null ideal S of finite index, then it has a definable, null ideal of finite index which contains S.
(ii) If R has a nilpotent ideal S of finite index, then it has a definable, nilpotent ideal of finite index which contains S.

Proof. (i) $Ann_R(S)$ is a (two-sided) ideal which contains S, and so it is the intersection of annihilators of only finitely many elements of S. Thus, it is definable. We conclude that $Ann_R(Ann_R(S))$ is a definable, null ideal containing S.
(ii) The proof will be by induction on the nilpotency class of S. If S is null, we are done by (i). For the induction step, let $n \geq 3$ be the nilpotency class of S, and let $I = Ann_R(S^{n-1})$ and $J = Ann_I(I)$. Then, I and J are ideals of R. Moreover, I contains S. Using additionally that S is of finite index in R, we see that I is definable (because it is the intersection of annihilators of finitely many elements of S^{n-1}), and so J is definable as well. Now, $(S + J)/J \cong S/(S \cap J)$ is of nilpotency class smaller than n (as $S^{n-1} \subseteq J$). Thus, by the induction hypothesis, there is a definable ideal K of R containing $S + J$ such that K/J is nilpotent. Since $J = Ann_I(I)$ is null, we

see that K is nilpotent. ∎

The same proofs yield the following remark (in 1(iii), one gets the conclusion by taking the closure of the group generated by all solvable, normal subgroups of finite index).

Remark 2.8 *1. Let (H, G) be a G-group.*
(i) If H has an abelian [normal] subgroup A of finite index, then it has a $*$-closed, abelian [normal] subgroup of finite index which contains A.
(ii) If H has a nilpotent [normal] subgroup N of finite index, then it has a $*$-closed, nilpotent [normal] subgroup of finite index which contains N.
(iii) If H has a solvable subgroup S of finite index, then it has a $*$-closed, solvable, normal subgroup K of finite index. If S is additionally normal in H, then K can be chosen to contain S.
2. Let (R, G) be a G-ring.
(i) If R has a null ideal S of finite index, then it has a $*$-closed, null ideal of finite index which contains S.
(ii) If R has a nilpotent ideal S of finite index, then it has a $*$-closed, nilpotent ideal of finite index which contains S.

In the last part of this section, according to the referee's suggestion, we try to reverse Theorem 2.1. It was rather easy to transfer properties of groups to the corresponding properties of rings by considering the appropriate groups of matrices with entries from the ring in question. For the other direction, it is not clear in general how to associate (in a definable way) a ring with a given group so that we can use the properties of this ring to conclude something about the group. So, this direction seems more difficult. It turns out, however, that the proof of [1, Theorem 4.1.1] yields the converse of Theorem 2.1(ii). In fact, [1, Theorem 4.1.1] tells us that the conjecture that each ω-categorical, stable group is abelian-by-finite is equivalent to the conjecture that each ω-categorical, stable ring is null-by-finite. So, our Theorem 2.1(ii) implies the 'left to right' direction in Baginski's result. Applying the proof of the other direction of Baginski's theorem, we will obtain the converse of Theorem 2.1(ii). Theorems 2.1(ii) and 2.9 imply Theorem 4.1.1 from [1] (using the fact that ω-categorical, stable groups are nilpotent-by-finite). In our formulations, however, there is no ω-categoricity and stability assumption.

Theorem 2.9 *If every nilpotent ring interpretable in $\mathcal{M}$ is null-by-finite, then every nilpotent group interpretable in $\mathcal{M}$ is abelian-by-finite.*

Proof. Suppose for a contradiction that there is a nilpotent group interpretable in $\mathcal{M}$ which is not abelian-by-finite. Let G be such a group of minimal nilpotency class. We claim that G is of class 2. Indeed, since $G/Z(G)$ is an interpretable group of smaller nilpotency class, there is a finite index subgroup H of G containing $Z(G)$ and such that $H/Z(G)$ is abelian. By Remark 2.5(i), we can find such an H which

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is additionally interpretable. Of course, H is of nilpotency class 2, and it is not abelian-by-finite. So, G is also of class 2.

Now, we follow the lines of Baginski's proof, eliminating the parts involving connected components. Write the group operations on $Z(G)$ and $G/Z(G)$ additively. Let R be the abelian group $G/Z(G) \times Z(G)$, equipped with the multiplication operation: $(aZ(G), x) \otimes (bZ(G), y) = (0, [a, b])$. This multiplication is associative (because applying it twice yields 0) and distributes over addition. Thus, R is a nilpotent ring interpretable in $\mathcal{M}$. Hence it has an ideal I of finite index which is null. Let $\overline{H}$ be the projection of I on the first coordinate, and let H the preimage of $\overline{H}$ under the quotient map $r : G \to G/Z(G)$. Then, H is a finite index subgroup of G, and for any $a, b \in H$ there are $x, y \in Z(G)$ such that $(aZ(G), x) \otimes (bZ(G), y) = (0, 0)$, i.e. $[a, b] = 0$, which means that H is abelian. $\blacksquare$

A similar proof yields the converse of Theorem 2.3(ii). In this result, we consider an extended definition of X^{teq} in which dividing by all $*$-closed (not necessarily $\emptyset$-closed) equivalence relations is allowed.

Theorem 2.10 *Let (X, G) be a compact G-space. Then, if every nilpotent ring $*$-closed in X^{teq} is null-by-finite, then every nilpotent group $*$-closed in X^{teq} is abelian-by-finite.*

The most essential in reversing Theorem 2.1(ii) was the ring structure obtained from a group of nilpotency class 2. This is related to the concept of the 'circle group of a ring' [11]. Skipping all details, on the universe of a nilpotent ring one can define a certain nilpotent group structure (called the circle group of the ring). On the other hand, as far as I know, it is conjectured that on every nilpotent group of class 2 one can define a nilpotent ring structure so that the group in question becomes the circle group of this ring (this was proved in many cases). If this conjecture was true in a 'definable way', one could prove the following statements, which are strengthenings of Theorems 2.9 and 2.10

Conjecture 2.11 *If every nilpotent ring interpretable in $\mathcal{M}$ is commutative-by-finite, then every nilpotent group interpretable in $\mathcal{M}$ is abelian-by-finite.*

Conjecture 2.12 *Let (X, G) be a compact G-space. Then, if every nilpotent ring $*$-closed in X^{teq} is commutative-by-finite, then every nilpotent group $*$-closed in X^{teq} is abelian-by-finite.*

3   Applications

First, we will use Theorems 2.1 and 2.3 to get structural results about ω-categorical rings and on small compact G-rings. Then, with Corollary 2.4, some relationships between several conjectures on small profinite groups will be obtained. In the last part, we prove a theorem which, in an ω-categorical context, allows one to deduce

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certain property of groups (virtual nilpotency of all solvable, definable groups) from the corresponding property of rings (virtual nilpotency of all interpretable rings).

Theorem 3.1 *An ω-categorical ring R satisfying NSOP (the negation of the strict order property) is nilpotent-by-finite.*

Proof. By Fact 1.2, we know that each group definable in R is nilpotent-by-finite, and thus, by Theorem 2.1(i), we get that R is nilpotent-by-finite. ∎

In fact, in this proof, we only need to know that each group definable in R is solvable-by-finite, which follows from the easier part of Macpherson's result (see [13, Section 4]). The above result is a strengthening of [2, Corollary to Theorem 3.6] which says that ω-categorical stable rings are nilpotent-by-finite. The proof in [2] is completely different from ours, and it uses the full NOP (the negation of the order property).

It was proved in [9] that each ω-categorical, supersimple ring is (finite-null)-by-null-by-finite. Using our Theorem 2.1(iii), this result follows immediately from the fact that ω-categorical, supersimple groups are (finite central)-by-abelian-by-finite [4].

Now, we will use Theorem 2.3 to get new information about small compact G-rings.

Theorem 3.2 *(i) A small, nm-stable compact G-ring is nilpotent-by-finite.*
(ii) A small, compact G-ring of finite $\mathcal{NM}$-rank is null-by-finite.

Proof. (i) Use Fact 1.3 and Theorem 2.3(i).
(ii) Use Fact 1.4 and Theorem 2.3(ii). ∎

[9, Theorem 3.4] tells us that small, nm-stable profinite rings are null-by-finite. It is worth mentioning that this result is an immediate consequence of our Corollary 2.4(ii) and the fact that small, nm-stable profinite groups are abelian-by-finite [17]. Similarly, using [15, Proposition 4.4] which says that small, m-normal profinite groups are abelian-by-finite (see [14, 15, 7] for the definition of m-normality), we get

Theorem 3.3 *Each small, m-normal profinite ring is null-by-finite.*

Theorem 3.2(ii) can be also obtained by application of [10, Proposition 2.8]. In order to see this, first we need to prove a lemma, which in fact will allow us to strengthen slightly Theorem 3.2(ii).

Lemma 3.4 *Let (R, G) be a small, compact G-ring. Assume there is an nm-generic g such that the left annihilator of g is open in R. Then R is null-by-finite.*

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Proof. There is an open (two-sided) ideal I of R contained in $Ann_R^{left}(g)$. Let $H = Stab(I) \leq G$. Then, by [5, Remark 5.10], H is a clopen subgroup of G and it has countable index in G. Let $\{g_i : i \in \omega\}$ be a set of representatives of left cosets of H in G. Then,

$$\bigcup_{i \in \omega} g_i H g = o(g) \subseteq_{nm} R.$$

Thus, there is $i \in \omega$ for which $g_i H g \subseteq_{nm} R$, and so $Hg \subseteq_{nm} R$. But, $Hg \subseteq Ann_R^{right}(I) \trianglelefteq R$. Hence, $Ann_R^{right}(I)$ is clopen in R. This implies that $I \cap Ann_R^{right}(I)$ is a clopen, null ideal of R. ∎

Theorem 3.5 *If (R, G) is a small compact G-ring, and either $\mathcal{NM}(R) < \omega$ or $\mathcal{NM}(R) = \omega^\alpha$ for some ordinal α, then R is null-by-finite.*

Proof. In the case $\mathcal{NM}(R) < \omega$, the idea of the proof is the same as in [9, Theorem 3.4]. Take an nm-generic orbit $o = o(a/\emptyset)$. For $g \in o$, define $R_g = \{(r, rg) : r \in R\}$ — a g-closed subgroup of $(R^+)^2$. By [10, Proposition 2.8], there is $b \underset{}{\overset{mm}{\lrcorner}} a$ and a b-closed subgroup K of $(R^+)^2$ commensurable with R_a. Take $a' \in o(a/b)$ such that $a' \underset{b}{\overset{mm}{\lrcorner}} a$. Then, $R_a \sim K \sim R_{a'}$, so $R_a \sim R_{a'}$ (where $\sim$ denotes commensurability). Let S be the projection of $R_a \cap R_{a'}$ on the first coordinate. Then, S is a finite index subgroup of R^+ which is contained in $Ann_R^{left}(a - a')$. So, $Ann_R^{left}(a - a')$ is clopen in R. On the other hand, we see that $a' \underset{}{\overset{mm}{\lrcorner}} a$, and so $a - a'$ is nm-generic. We finish using Lemma 3.4.

Now, consider the case $\mathcal{NM}(R) = \omega^\alpha$. In this case, we will often use Lascar inequalities for groups (see [10, Fact 1.1]). To be precise, we will be working in a slightly more general context in which R is a $*$-closed ring in X^{teq}, where (X, G) is a small compact G-space. By Theorem 3.2(i), R is nilpotent-by-finite. Hence, by Remark 2.8, we can assume that R is nilpotent. We argue by induction on the nilpotency class n of R. If $\mathcal{NM}(Ann_R(R)) = \omega^\alpha$, then $Ann_R(R)$ has finite index in R, and we are done. If $\mathcal{NM}(Ann_R(R)) < \omega^\alpha$, then $\mathcal{NM}(R/Ann_R(R)) = \omega^\alpha$, so, by the induction hypothesis, we get that $R/Ann_R(R)$ is null-by-finite. By Remark 2.8, we can assume that it is null. Take an nm-generic $g \in R$, and consider a homomorphism $f_g : R^+ \to R^+$ given by $f_g(r) = rg$. Since $R/Ann_R(R)$ is null, $Im(f_g) \subseteq Ann_R(R)$. Thus, $\mathcal{NM}(ker(f_g)) = \omega^\alpha$, and so $Ann_R^{left}(g)$ is clopen in R. We finish using Lemma 3.4. ∎

Conjecture 3.6 *A small, nm-stable compact G-ring is null-by-finite.*

A similar example to Example A of [5] shows that the above conjecture strongly fails without the assumption of nm-stability.

Example Let S_∞ be the group of all permutations of ω. It acts on the ring $R := \mathbb{Z}_p^\omega$ permuting coordinates. Arguing as in [5, Example A], we get that (R, S_∞) is a small, compact S_∞-ring which is not nilpotent-by-finite.

Let us notice one more corollary of Lemma 3.4.

Corollary 3.7 *Let (R, G) be a small compact G-ring. If R is countable-by-null-by-countable, then it is null-by-finite.*

Proof. There is a countable index subring R_1 of R and a countable ideal I_1 of R_1 such that R_1/I_1 is null. So, by the Baire category theorem, $R_1 \subseteq_{nm} R$. Hence, there is $g \in R_1$ which is nm-generic in R. We see that Rg is countable, which implies that $Ann_R^{left}(g)$ is clopen in R. We finish using Lemma 3.4. ∎

The above results yield a quite good understanding of small, nm-stable compact G-rings. A more general problem is to classify small compact G-groups and small compact G-rings which are not necessarily nm-stable. One could try at least to describe small compact G-rings without nilpotent elements (as was suggested by the referee). Moreover, one could try to find a description of the structure of small [nm-stable] Polish G-rings (i.e. G-rings (R, G), where R is Polish) and to search for examples of such rings.

In the next part of this section, using Corollary 2.4, we will obtain surprising relationships between some conjectures on small profinite groups.

Recall the main conjecture on small profinite groups proposed by Newelski.

Conjecture 3.8 *Each small profinite group is abelian-by-finite.*

The following three intermediate conjectures are still open.

Conjecture 3.9 *For each small profinite group (H, G):*
(A) H is solvable-by-finite,
(B) if H is solvable-by-finite, then H is nilpotent-by-finite,
(C) if H is nilpotent-by-finite, then H is abelian-by-finite.

For small profinite rings, we have the following conjectures.

Conjecture 3.10 *Each small profinite ring is null-by-finite.*

Conjecture 3.11 *For each small profinite ring (R, G):*
(A') R is nilpotent-by-finite,
(B') if R is nilpotent-by-finite, then R is null-by-finite.

In fact, the proof of [9, Remark 3.2] shows that (B') implies (A'). A step toward the proof of (A') was done in [9], namely each small profinite ring was shown to be (nil of finite nil exponent)-by-finite. We will need [9, Theorem 3.5], namely:

Fact 3.12 *Conjecture (A') restricted to commutative rings implies Conjecture (B).*

Using this fact together with Corollary 2.4, we get the following

Corollary 3.13 *(i) (A) implies (A') implies (B).*
(ii) (C) implies (B') implies (A') implies (B).

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This corollary is surprising, because it implies that in order to show Conjecture 3.8, it is enough to prove Conjectures (A) and (C) (although Conjecture (B) has not been proved). One could try to go further and to show that Conjecture 3.8 reduces to showing Conjecture (A). A possible way to do that could be through the following conjecture.

Conjecture 3.14 *(B) implies (B').*

Indeed, if we knew that (B) implies (B'), then by Remark 2.8 and Theorem 2.10, we would get that (B) implies (C). Thus, by virtue of Corollary 3.13(i), we would conclude that (A) implies (B) and (C).

Analyzing the proof of Fact 3.12, one gets that all the observations below Conjecture 3.8 are true in the class of small profinite groups and rings satisfying any extra assumption which is preserved under taking $*$-closed groups and rings in imaginary sorts. For example, nm-stability and m-normality are preserved under taking $*$-closed groups, and so [9, Theorem 3.4] and our Theorem 3.3 follow from Corollary 3.13 and the results saying that small, nm-stable [or m-normal] profinite groups are abelian-by-finite.

As was mentioned in Section 2, it is not clear in a general context how to transfer various properties from definable rings to definable groups. Theorem 2.9 was an exception. Another result of this kind is Fact 3.12, but the context in here is restricted to small profinite groups. We finish the paper with a similar result for ω-categorical groups.

Theorem 3.15 *Let $\mathcal{M}$ be an ω-categorical structure. Suppose every group definable in $\mathcal{M}$ has a connected component and every commutative ring interpretable in $\mathcal{M}$ is nilpotent-by-finite. Then every solvable group definable in $\mathcal{M}$ is nilpotent-by-finite.*

Before the proof, recall that a group G definable in a monster model $\mathfrak{C}$ is said to have a connected component, G^0, if the intersection of all definable subgroups of finite index, denoted by G^0, has bounded index; equivalently, it coincides with an intersection of only boundedly many definable subgroups of finite index (so it is type-definable). In the ω-categorical context, this amounts to saying that there exists the smallest definable subgroup of finite index (which equals G^0). We take the last statement as the definition of the existence of connected components in non-saturated ω-categorical groups.

Let us notice that the extra assumption about the existence of the connected components is satisfied for example under NIP. Indeed, if a monster model $\mathfrak{C} \succ \mathcal{M}$ has NIP and G is a group definable in $\mathfrak{C}$, then G^{00} (the smallest type-definable subgroup of bounded index) exists and is $\emptyset$-invariant, and so $\emptyset$-definable by ω-categoricity. Thus, $G^{00} = G^0$ is the smallest definable subgroup of finite index. Moreover, from [8], ω-categorical rings satisfying NIP are nilpotent-by-finite. So, Theorem 3.15 shows that solvable, ω-categorical groups with NIP are nilpotent-by-finite.

Notice also that by Theorem 3.15, all the relationships between conjectures on small profinite groups and rings formulated in Fact 3.12 and Corollary 3.13 have

their counterparts in the context of ω-categorical structures satisfying any extra assumption which is preserved under interpretatability and which implies (together with ω-categoricity) that all definable groups have connected components.

Another remark is that from the proof below, it follows that in Theorem 3.15 it is enough to assume the existence of centralizer connected components (instead of connected components) for all interpretable groups.

Now, we turn to the proof of Theorem 3.15. In fact, we will modify the proof of [13, Theorem 1.2] (which says that solvable, ω-categorical groups with NSOP are nilpotent-by-finite), replacing the part of the argument involving NSOP by a new argument using virtual nilpotency of a certain interpretable ring. We could also give an argument based on the Baur-Cherlin-Macintyre proof that ω-categorical, ω-stable groups are abelian-by-finite [3] (similarly to the proof of Fact 3.12). We prefer to give a proof based on [13], because it leads to a question, a positive answer to which would allow us to drop the assumption about the existence of connected components in Theorem 3.15.

First, recall Proposition 3.4 from [13], where we skip some unnecessary assumptions.

Fact 3.16 *Let G be a countable, ω-categorical group with a normal subgroup V which is a vector space over $F := GF(p^a)$ (p is a prime number). Let $H := G/V$, and assume H has no elements of order p. Suppose V is a sum of finite dimensional over F (so finite) FH-modules. Then:*
(i) V is a direct sum of finite dimensional over F, irreducible FH-modules,
(ii) $[H : C_H(V)] < \omega$.

Proof. For (i) see the proof of (i) in [13, Proposition 3.4].
(ii) We simplify a bit the proof of (ii) in [13, Proposition 3.4]. For $S \subseteq V$ and $K \subseteq H$

$$H_S := \{h \in H : (\forall s \in S)(hs = s)\} \text{ and } Fix(K) := \{v \in V : (\forall k \in K)(kv = v)\}.$$

By (i), $V = \bigoplus_{i \in \omega} V_i$ for some finite dimensional (over F) FH modules V_i.

We construct a sequence $r_0, r_1, \ldots$ of pairwise distinct natural numbers and elements $v_0 \in V_{r_0}, v_1 \in V_{r_1}, \ldots$ in the following way. r_0 and $v_0 \in V_{r_0}$ are chosen arbitrarily. Suppose $r_0, \ldots, r_n$ and $v_0, \ldots, v_n$ have been chosen. If for every $r \notin \{r_0, \ldots, r_n\}$, $V_r \subseteq Fix(H_{v_0, \ldots, v_n})$, we stop the construction. If not, we choose any r_{n+1} and $v_{n+1} \in V_{r_{n+1}}$ so that $v_{n+1} \notin Fix(H_{v_0, \ldots, v_n})$. We claim that this construction stops after finitely many steps. Indeed, otherwise $v_0, v_0 + v_1, v_0 + v_1 + v_2, \ldots$ lie in finite but increasingly large conjugacy classes of G, contradicting the Ryll-Nardzewski theorem. Hence, there is n such that $\bigoplus \{V_i : i \notin \{r_0, \ldots, r_n\}\} \subseteq Fix(H_{v_0, \ldots, v_n})$. Thus, $V = Fix(H_{V_{r_0} \cup \cdots \cup V_{r_n}})$, i.e. $H_{V_{r_0} \cup \cdots \cup V_{r_n}} = C_H(V)$. On the other hand, since $V_{r_0} \cup \cdots \cup V_{r_n}$ is finite and invariant under H, $[H : H_{V_{r_0} \cup \cdots \cup V_{r_n}}] < \omega$. Therefore, $[H : C_H(V)] < \omega$. ∎

Now, we will prove a suitable variant of [13, Corollary 3.5].

Corollary 3.17 *Let $\mathcal{M}$ be a countable, ω-categorical structure and G be a group interpretable in it. Suppose G has a normal subgroup V interpretable in $\mathcal{M}$ which is a vector space over $F := GF(p^a)$ (p is a prime number). Assume that each commutative ring interpretable in $\mathcal{M}$ is nilpotent-by-finite. Let $H := G/V$, and suppose H is nilpotent and has no elements of order p. Assume that each subgroup of H interpretable in $\mathcal{M}$ has a connected component. Then V is a direct sum of finite dimensional over F, irreducible FH-modules, and $[H : C_H(V)]$ is finite.*

Proof. We will argue by induction on the nilpotency class of H. First, we will reduce the induction step to the base step. Suppose H is of nilpotency class n, and the conclusion holds whenever the nilpotency class is smaller than n. Then, $Z_n(H) = H$. Put $H_1 = Z_{n-1}(H)$. By the induction hypothesis, $[H_1 : C_{H_1}(V)]$ is finite. Moreover, both H_1 and $C_H(V)$ are normal subgroups of H, so $C_{H_1}(V)$ is also normal in H. Since $[H, H] \leq H_1$, we conclude that all elements of $H/C_{H_1}(V)$ have centralizers of finite index. So, replacing H by its connected component multiplied by $C_{H_1}(V)$, we can assume that $H/C_{H_1}(V)$ is abelian (Fact 3.16 assures that we do not loose generality modifying H in this way). Notice that it is enough to show the desired conclusion for $H/C_{H_1}(V)$ acting on V by conjugation. Thus, everything has been reduced to the base step, i.e. to the case when H is abelian.

As in [13], we define W as the sum of all finite dimensional FH-submodules of V. This is an interpretable in $\mathcal{M}$ subspace of V invariant under H. By Fact 3.16, it is enough to show that $W = V$. Suppose for a contradiction that $W \subsetneq V$, and put $\overline{V} = V/W$. Then, exactly as in [13], we get the following Claim.

Claim The FH-module $\overline{V}$ has no non-trivial, finite dimensional FH-submodules.

The rest of the proof differs from [13]. Choose a non-trivial $v \in \overline{V}$, and put $V_0 = Lin_F(Hv)$ (the subgroup of $\overline{V}$ generated by all elements hv, $h \in H$). By ω-categoricity, V_0 is interpretable. Let R be the ring of endomorphisms of V_0 generated by H. By the commutativity of H, we get that R is commutative and interpretable in $\mathcal{M}$ because each $r \in R$ is determined by its value on v. So, R has a nilpotent ideal I of finite index m; say $r_1, \ldots, r_m$ are representatives of all cosets of I in R. By the claim, V_0 is infinite, and so R and I are infinite as well. Let $k \geq 2$ be the nilpotency class of I, i.e. the smallest number for which $I^k = \{0\}$. Choose any non-trivial $i \in I^{k-1}$. Then, $i(v) \neq 0$ and $Hi(v) \subseteq Ri(v) = \{r_1 i(v), \ldots, r_m i(v)\}$. Thus, $Lin_F(Hi(v))$ is a non-trivial, finite dimensional (over F) FH-submodule of $\overline{V}$ the existence of which contradicts the claim. ∎

In order to prove Theorem 3.15, we can assume that $\mathcal{M}$ is countable. Having Corollary 3.17, the proof of [13, Theorem 1.2] from page 490 of [13] (which does not use NSOP anymore) goes through in our context and completes the proof of Theorem 3.15.

The assumption about the existence of connected components in Theorem 3.15 was only used in order to reduce the induction step to the base step. We needed

this reduction (to the case when H is abelian) in order to make sure that the ring R defined later in the proof is interpretable and commutative. So, one can easily show that a positive answer to the following question would allow us to drop the extra assumption about connected components, but under a stronger assumption that all (not necessarily commutative) rings interpretable in $\mathcal{M}$ are nilpotent.

Question 3.18 *We work in a countable, ω-categorical structure $\mathcal{M}$. Let H be a definable group of nilpotency class 2 and with finite center. Suppose H acts definably and by automorphisms on a definable, abelian group V [which is a vector space over $GF(p^a)$, and H has no elements of order p]. Take $v \in V$, and put $V_0 = \langle Hv \rangle \leq V$. Let R be the ring of endomorphisms of V_0 generated by H. Is it true that R is interpretable in $\mathcal{M}$?*

References

[1] P. Baginski, *Stable $\aleph_0$-categorical algebraic structures*, Ph.D. thesis, Berkeley, 2009.

[2] J. Baldwin, B. Rose, *$\aleph_0$-categoricity and stability of rings*, J. Alg. 45, 1-16, 1977.

[3] W. Baur, G. Cherlin, A. Macintyre, *Totally categorical groups and rings*, J. Alg. 57, 407-440, 1979.

[4] D. Evans, F. O. Wagner, *Supersimple ω-categorical groups and theories*, J. Symb. Logic 65, 767-776, 2000.

[5] K. Krupiński, *Some model theory of Polish structures*, Trans. Amer. Math. Soc. 362, 3499-3533, 2010.

[6] K. Krupiński, *Fields interpretable in rosy theories*, Israel Journal of Mathematics 175, 421-444, 2010.

[7] K. Krupiński, *Generalizations of small profinite structures*, Journal of Symbolic Logic 75, 1147-1175, 2010.

[8] K. Krupiński, *On ω-categorical groups and rings with NIP*, submitted.

[9] K. Krupiński, F. Wagner, *Small profinite groups and rings*, J. Alg. 306, 494-506, 2006.

[10] K. Krupiński, F. Wagner, *Small, nm-stable compact G-groups*, submitted.

[11] R. L. Kruse, D. T. Price, *Nilpotent rings*, Gordon and Breach Science Publishers, New York, 1969.

[12] J. Lewin, *Subrings of finite index in finitely generated rings*, J. Alg. 5, 84-88, 1967.

[13] H. D. Macpherson, *Absolutely ubiquitous structures and $\aleph_0$-categorical groups*, Quart. J. Math. Oxford 39, 483-500, 1988.

[14] L. Newelski, *Small profinite structures*, Trans. Amer. Math. Soc. 354, 925-943, 2001.

[15] L. Newelski, *Small profinite groups*, Journal of Symbolic Logic 66, 859-872, 2001.

[16] L. Ribes, P. Zalesskii, *Profinite groups*, Ergeb. Math. Grenzgeb.40, Springer, Berlin, 2000.

[17] F. O. Wagner, *Small profinite m-stable groups*, Fund. Math. 176, 181-191, 2003.

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