SMALL PROFINITE GROUPS AND RINGS

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Abstract. A small profinite ring has an open nil ideal of finite nil exponent. We also discuss various conjectures about small profinite rings, and give examples illustrating them.

1. Introduction

Following Newelski [7, 8], who introduced small profinite structures and groups, in this paper we shall consider profinite rings regarded as profinite structures, namely an inverse limit R of finite rings, together with a closed subgroup $Aut^*(R)$ of the group of automorphisms respecting the inverse system. Such a structure will be called *small* if $Aut^*(R)$ has only countably many orbits on n-tuples for all $n < \omega$. Newelski conjectured:

Conjecture 1. *A small profinite group has an open abelian subgroup.*

For the restricted class of m-stable profinite groups (see the next section for a definition) the conjecture was proven by the second author [12]; it is still open in the general case. Note that the first author [4] established that an abelian profinite group of finite exponent which is the inverse limit of a system indexed by ω and considered with the group of all automorphisms respecting this system is small and m-stable.

It seems that, similarly to the proof of the Baur-Cherlin-Macintyre Theorem [9, Theorem 1.13] on ω-stable ω-categorical groups, it is necessary to consider certain rings of endomorphisms interpretable in small profinite groups, as was done for instance in the m-stable case [7, Lemma 3.4]. That was our initial motivation for considering small profinite rings, but of course they have an interest in their own right.

Small profinite structures are remarkably similar to ω-categorical stable structures, or even more so, ω-categorical simple structures.

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In this context, m-stability corresponds to superstability (actually ω-stability), or supersimplicity. Newelski's $\mathcal{M}$-gap conjecture [6] is then the analogue of the Cherlin-Harrington-Lachlan theorem [1] on finiteness of rank for ω-stable ω-categorical structures; its proof for groups [12] mimicks the ω-categorical supersimple case [2]. One might hence be tempted to conjecture at least:

Conjecture 2. *A small profinite group has an open nilpotent subgroup.*

Conjecture 3. *A small soluble profinite group has an open nilpotent subgroup.*

In the context of rings, these conjectures become (recall that a ring is *null* if multiplication is trivial):

Conjecture 4. *A small profinite ring has an open null ideal.*

Conjecture 5. *A small profinite ring has an open nilpotent ideal.*

Small profinite groups and rings occur naturally in model theory as quotients G/G^0 and R/R^0, where G and R are a small group and ring, and G^0 and R^0 are the intersection of all $\emptyset$-definable normal subgroups/ideals of finite index. Nevertheless, there are only few explicit examples, namely some classes of small profinite groups [3, 4]. Conjectures 1 and 4 state more or less that they are all there is. In particular, if Conjecture 4 is true, the study of small profinite rings becomes obsolete, but we obtain information about small profinite groups, e.g. that Conjecture 3 holds. If on the other hand it is false, there will be interesting examples of small profinite rings.

After a section on preliminaries, we shall show in section 3 that small profinite rings have an open nil ideal of finite nilexponent. We shall also observe that Conjecture 4 holds in the m-stable case. In section 4 we shall prove various reductions for Conjecture 5. Finally we shall discuss examples in section 5. In particular, any expansion of a small profinite commutative group to a ring with open annihilator is shown to be small profinite (recall that the annihilator of a ring is the set of elements $r \in R$ with $rR = Rr = 0$). However, we also give an example of a small profinite ring whose annihilator is nowhere dense.

2. Notation and preliminaries

The rings considered may or may not have a unit, as needed. All ideals will be two-sided (unless qualified as left or right).

The following fact [10, Proposition 5.1.2] gives a characterization of when a topological ring is profinite.

Fact 2.1. *Let R be a topological ring (so, in particular, a Hausdorff topological space). Then the following conditions are equivalent:*

(1) *R is profinite, i.e. the inverse limit of finite rings.*
(2) *R is compact.*
(3) *R is compact and totally disconnected.*
(4) *R is compact and there is a basis of open neighbourhoods of* 0 *consisting of open ideals.*

Suppose we have a profinite ring R which is the inverse limit of a distinguished inverse system, or equivalently R is a compact topological ring with a distiguished basis of open neighbourhoods of 0 consisting of clopen ideals.

Definition 2.2. A *profinite ring regarded as profinite structure* is a pair of the form $(R, Aut^*(R))$, where $Aut^*(R)$ is a closed subgroup of the group $Aut^0(R)$ of all automorphisms of R respecting the inverse system defining R (equivalently, $Aut^*(R)$ is a closed subgroup of the group of all automorphisms of R fixing setwise the clopen ideals from the distiguished basis of open neighbourhoods of 0). The group $Aut^*(R)$ is called the *structural group* of R and $Aut^0(R)$ the *standard structural group* of R.

For the general definition of a profinite structure see [8]. We shall always assume that the structure is infinite, with a countable inverse system.

From now on all profinite rings will be regarded as profinite structures so we will skip the phrase 'regarded as profinite structure'. Also, we will often write R instead of $(R, Aut^*(R))$. Let $A \subseteq R$ be finite. By an *A-invariant* subset of R^n we mean a subset invariant under $Aut^*(X/A)$, and by an *A-closed* subset we mean a closed and A-invariant subset of R^n; if we do not want to specify A, we write **-invariant* or **-closed*. For $a \in R^n$ and $A \subseteq X$ we define $o(a/A) = \{f(a) : f \in Aut^*(X/A)\}$, the *orbit* of a over A.

The following remark follows directly from the definition.

Remark 2.3. *A profinite ring R has a descending chain $(I_n : n < \omega)$ of open $\emptyset$-invariant ideals forming a basis of open neigbourhoods of* 0 *(hence with trivial intersection).*

Definition 2.4. A profinite ring is *small* if there are only countably many orbits on finite tuples over $\emptyset$.

Definition 2.5. A profinite ring is *m-stable* if there is no infinite sequence $A_1 \subseteq A_2 \subseteq \cdots$ of finite subsets of R and $a \in R$ such that $o(a/A_{i+1})$ is nowhere dense in $o(a/A_i)$ for every i.

Definition 2.6. Let R be a profinite ring (or more generally a profinite structure). The structure R^{eq} is obtained from R in the following way. For any $\emptyset$-closed equivalence relation on some R^n we adjoin a new (imaginary) sort $R_E = R^n/E$. R is identified with $R_=$. Then $Aut^*(R)$ acts continuously on every sort, and hence on R^{eq} (with the disjoint union topology). If R is small, then every sort (with the induced structural group) is again a profinite structure, and R^{eq} is a many-sorted profinite structure.

By $acl(A)$ we denote the set of elements of R^{eq} which have finite orbits over A.

We say that $g \in R$ is *generic* over A if $o(g/A)$ is open. We say that a is *m-independent* from b over A (denoted $a \overset{m}{\perp}_A b$) if the orbit $o(a/Ab)$ is open in $o(a/A)$. Otherwise a is *m-dependent* on b over A (written $a \overset{m}{\not\perp}_A b$). The relation $\overset{m}{\perp}$ enjoys properties analogous to forking independence in stable theories.

The following two facts due to Newelski [7] are fundamental:

Fact 2.7. *Let G be a small profinite group, and A a finite subset. An A-invariant subgroup of G is A-closed. The group generated by any family of A-invariant sets is A-closed, and generated in finitely many steps from finitely many sets. There is no infinite increasing chain of A-invariant subgroups of G. In particular all characteristic subgroups of G are $\emptyset$-closed.*

Corollary 2.8. *Let G be a small profinite group, and A a finite subset. Then the intersection $G \cap acl(A)$ is finite. In particular G is locally finite, and if G is a profinite ring, it is locally finite as a ring.*

The following remark shows that any ring interpretable in a small profinite structure is a small profinite as a ring.

Remark 2.9. *Let U be a small profinite structure and A a finite subset of U. If a ring R is A-closed in U^{eq} (i.e. R and the graphs of addition and multiplication are A-closed), then R is profinite with topology inherited from U^{eq} and it has a basis of open neighbourhoods of 0 consisting of open A-closed ideals. In this way $(R, Aut^*(U^{eq}/A) \restriction R)$ is a profinite ring regarded as profinite structure. Moreover, if U is small, then so is R.*

Proof: Proposition 1.4 of [7] tells us that the the counterpart of this remark for R being a group (insted of ring) holds. Apply this to the additive group of R. We get that $(R, +)$ is a profinite group with topology iherited from U^{eq} and there is a basis $\{H_n : n \in \omega\}$ of open neighbourhoods of 0 consisting of A-closed subgroups of $(R, +)$. Moreover, the ring multiplication is *-closed, hence continuous.

For every n, since $R0R = \{0\} \leq H_n$, compactness implies that there is m such that $RH_mR \leq H_n$. Put $I_n = \{r \in H_n : RrR \leq H_n\}$. Then I_n is an A-invariant ideal containing $H_m \cap H_n$, hence open. Clearly the $(I_n : n < \omega)$ form a basis of open neighbourhoods of 0 consisting of A-closed ideals. □

Let us recall now several classical definitions from ring theory. An element r of a ring R is *nilpotent* of *nilexponent* n if $r^n = 0$ and n is the smallest number with this property. The ring is *nil* (of *nilexponent* n) if every element is nilpotent (of nilexponent $\leq n$, and there is an element of nilexponent n). The ring is *nilpotent* of *class* n if $r_1 r_2 \cdots r_n = 0$ for all $r_1, r_2, \dots, r_n \in R$, and n is the smallest number with this property. An element r is *null* if $rR = Rr = \{0\}$. A ring is *null* if all its elements are.

Definition 2.10. The *Jacobson radical* $J(R)$ of a ring R is the set of all elements $r \in R$ such that $1 - xr$ is left-invertible for all $x \in R$.

Clearly $J(R)$ is characteristic. It is a (two-sided) ideal equal to the intersection of all maximal left ideals, and also to the intersection of all maximal right ideals. The Jacobson radical contains all one-sided nil ideals; moreover $J(R/J(R)) = J(R)$.

3. Small profinite rings

We start from our main theorem which is a first step towards the proof of Conjecture 5.

Theorem 3.1. *A small profinite ring has an open nil ideal of finite nilexponent.*

Proof: We shall show first that $R/J(R)$ is finite. So consider a generic (open) orbit o of R, and an open $\emptyset$-invariant ideal I such that $o + I \subseteq o$. Fix $i \in I$ and $a \in o$.

Claim. *There is $m > 0$ such that $a(ia)^m = 0$.*

Proof: By local finiteness there are $n > m > 0$ such that $(ia)^n = (ia)^m$. Since $a(ia)^{n-m} \in I$, there is an automorphism σ of R with $\sigma(a) = a - a(ia)^{n-m}$. Let I' be any $\emptyset$-invariant open ideal. Then σ maps the left ideal $Ra + I'$ to the left ideal $R(a - a(ia)^{n-m}) + I' \leq Ra + I'$. But since this is an ideal of finite index, we must in fact have equality, and there is $r \in R$ with $a \in r(a - a(ia)^{n-m}) + I'$. Therefore

$$a(ia)^m \in r(a(ia)^m - a(ia)^n) + I' = r(a(ia)^m - a(ia)^m) + I' = I'.$$

Since I' was arbitrary and the intersection of all open $\emptyset$-invariant ideals is trivial, $a(ia)^m = 0$. □

Hence $(ia)^{m+1} = 0$; since $i \in I$ was arbitrary, Ia is nil. So $Ia \leq J(R)$. But for $a \in o$ we also have $a + i \in o$, so $i(a + i) \in J(R)$, and $i^2 = i(a + i) - ia \in J(R)$. Hence $I/J(R)$ is nil, and I is contained in $J(R)$. It follows that $J(R)$ is open.

Next, let $a \in J(R)$. By local finiteness again there are $n > m > 0$ with $a^n = a^m$. Then $a^m(1 - a^{n-m}) = 0$; but $1 - a^{n-m}$ is invertible, as $a \in J(R)$. Hence $a^m = 0$ and $J(R)$ is nil.

Now if R is commutative, we may consider an open nil orbit o. Then there is $n < \omega$ such that $a^n = 0$ for all $a \in o$. But there is an open invariant ideal I with $o + I = o$, so $I \subseteq o - o$. By commutativity every element of $o - o$ has nilexponent at most $2n - 1$, so I has finite nilexponent.

In the non-commutative case, consider an open nil orbit o in $J(R)$ of nilexponent n, and an open ideal $I \leq J(R)$ with $o + I \subseteq o$.

Claim. *$x^{n-1}yx^{n-1} = 0$ for all $x \in o$ and $y \in I$.*

Proof of Claim: We have $x + yx^{n-1} \in o$; since $x^n = 0$, we obtain

$$\begin{aligned} 0 &= (x + yx^{n-1})^n = x^{n-1}yx^{n-1} + x^{n-2}(yx^{n-1})^2 + \cdots + (yx^{n-1})^n \\ &= (1 + z)x^{n-1}yx^{n-1} \end{aligned}$$

for some $z = x^{n-2}y + x^{n-3}yx^{n-1}y + \cdots \in J(R)$. As $1 + z$ is invertible, $x^{n-1}yx^{n-1} = 0$. □

Consider the set $X = \{x \in J(R) : \forall y \in I\, xyx = 0\}$. Since I and $J(R)$ are ideals, we have $X = RXR$. Let H be the ideal additively generated by X. By Fact 2.7 there is $k < \omega$ such that every element of H is a sum of at most k elements in X. But if $h = x_1 + \cdots + x_k$ with $x_1, \ldots, x_k \in X$, then for $y_1, \ldots, y_k \in I$

$$hy_1hy_2 \cdots hy_kh = \sum_{1 \leq i_0, i_2, \ldots, i_k \leq k} x_{i_0}y_1x_{i_1}y_2 \cdots y_kx_{i_k} = 0,$$

as every summand in the expanded product contains a factor x_izx_i for some $i \leq k$ and $z \in I$. Hence IhI is nilpotent of class at most $k + 1$.

By Fact 2.7 the ideal $Z_0 := \{r \in J(R) : IrI \text{ is nilpotent}\}$ is $\emptyset$-closed, and the nilpotency class of IrI is bounded for $r \in Z_0$ by some $c_0 < \omega$. Similarly, if the ideal Z_i is $\emptyset$-closed, then

$$Z_{i+1} := \{r \in J(R) : IrI/Z_i \text{ is nilpotent}\}$$

is $\emptyset$-closed, and the nilpotency class of IrI/Z_i is bounded by some $c_{i+1} < \omega$ for all $r \in Z_{i+1}$. Now $x^{n-1} \in Z_0$ for all $x \in o$, so $x^{n-2} \in Z_1$, etc., and $x \in Z_{n-2}$. It follows that Z_{n-2} is open, and has nilexponent at most $c_0 \cdot c_1 \cdots c_{n-2}$. As $J(R)/Z_{n-2}$ is finite, $J(R)$ has bounded nilexponent. □

Clearly, if R is commutative and $r \in R$ nil, then the principal ideal (r) is nilpotent, of class the nilexponent of r. We conjecture:

Conjecture 6. *In a small profinite ring, the principal ideal generated by any element in the Jacobson radical is nilpotent.*

Remark 3.2. *Theorem 3.1 does not help us to prove Conjecture 4, as we can easily reduce it to the nilpotent and commutative case.*

Proof: Given a ring R, define a ring structure on $P = R \times R \times R$ with product addition and multiplication given by

$$(a_1, b_1, c_1) \cdot (a_2, b_2, c_2) = (0, 0, a_1 b_2 + a_2 b_1).$$

Then P is commutative and nilpotent of class ≤ 3; with the natural induced structural group P is profinite small iff R is. Moreover, P has an open null ideal iff R does. $\square$

We now turn to the proof of Conjecture 4 in the m-stable case.

Theorem 3.3. *A small profinite m-stable ring R has an open null ideal.*

Proof: Considering R as an abelian subgroup with additional structure, [12, Theorem 18] tells us that it has finite $\mathcal{M}$-rank, and [12, Theorem 15] that every *-closed group is commensurable with one defined over a finite tuple in $\mathrm{acl}(\emptyset)$. So consider an open orbit o, and for $a \in o$ the subgroup $R_a := \{(r, ra) : r \in R\} \leq (R^+)^2$. Let $c \in \mathrm{acl}(\emptyset)$ be a finite tuple such that there is a c-closed subgroup commensurable with R_a. Then R_a and $R_{a'}$ are commensurable for any $a' \in o(a/c)$ (which we choose m-independent of a); since c is algebraic, $o(a/c)$ is still open.

The projection of $R_a \cap R_{a'}$ to the first coordinate has finite index in R^+, and hence contains an open $\emptyset$-invariant ideal I. For $r \in I$ we have $ra = ra'$, so $r(a - a') = 0$. Since a and a' are m-independent generic, $a - a'$ is again generic, and $o(a - a')$ contains a coset of an open $\emptyset$-invariant ideal I'. By invariance, $Ia'' = \{0\}$ for all $a'' \in o(a - a')$, so $II' = \{0\}$. Thus $I \cap I'$ is an open null ideal. $\square$

Let us remark in passing that a similar proof works for ω-categorical supersimple rings (see [11] for definitions and results concerning simplicity):

Theorem 3.4. *An ω-categorical supersimple ring has an ideal of finite index which is (finite null)-by-null.*

Proof: As in the proof of Theorem 3.3, using [11, Propositions 6.2.42 and 6.2.40] instead of [12, Theorems 18 and 15], we obtain independent generic a, a' such that R_a and $R_{a'}$ are commensurable, and hence a generic element $a - a'$ whose left annihilator has finite index in R. So the set of all $r \in R$ whose left annihilator has finite index in R is definable and generic; note that in fact it is a (two-sided) ideal. By symmetry the ideal of elements with generic right annihilator is also definable and generic; the intersection of the two yields a definable ideal H of finite index. For independent $h, h' \in H$ we have

$$hh' \in \mathrm{acl}(h) \cap \mathrm{acl}(h') = \mathrm{acl}(\emptyset),$$

it follows that $H^2 \subseteq \mathrm{acl}(\emptyset)$, where H^2 is the subring (hence ideal) generated by all products h_1h_2 for $h_1, h_2 \in H$. So H^2 is a finite ideal, and its two-sided annihilator is an ideal I of finite index in R. Put $Z = I \cap H^2$, a finite ideal of R null in $I \cap H$; clearly $I \cap H$ is null modulo Z and has finite index in R. $\square$

At the end of this section we discuss a connection between Conjectures 3 and 5. As mentioned in the introduction, the second author proved Conjecture 1 for m-stable profinite groups [12]. But an important step in this proof was done by Newelski [7, Theorem 3.3], who showed Conjecture 3 in the m-stable case. We shall now adapt his proof to show:

Theorem 3.5. *Conjecture 5 for commutative rings implies Conjecture 3.*

Proof: Newelski's proof consists of three steps: [7, Lemma 3.4], [7, Lemma 3.5], and the final proof of [7, Theorem 3.3]. He uses m-stability only in the first two steps. However, it is easy to check that this assumption is irrelevant in [7, Lemma 3.5]. So to apply Newelski's proof in our situation it is enough to prove [7, Lemma 3.4] for profinite groups which are not necessarily m-stable:

Lemma 3.6. *Let U be a small profinite structure and A a finite subset of U. Assume that D, H are infinite A-closed abelian groups in U^{eq}. Assume that D acts definably on H as a group of automorphisms. Then some open subgroup of D fixes pointwise an infinite subgroup of H.*

Proof: The first part of the proof is the same as in [7] so we only sketch the proof (see [7] for the details). Namely, let $(D_n, n < \omega)$, be a sequence of open A-closed subgroups of D generating the topology on D. Then we can assume that $\bigcup_n Fix(D_n)$ is finite and, as a consequence, that the set of these $a \in H$ for which Da is finite forms a finite D-invariant subgroup H_0 of H. Hence there is $a \in H$ such that Da is infinite. Let H_1 be a subgroup of H generated by Da. Then H_1 is $(A \cup \{a\})$-closed. Let $R \subseteq End(H_1)$ be the ring generated by the elements of D, regarded as endomorphisms of H_1. Each element of R

is determined by its value on a. Then R is a commutative ring interpretable (as a ring) in U^{eq} over $A \cup \{a\}$, i.e. R can be identified with an $(A \cup \{a\})$-closed ring in U^{eq}. By Remark 2.9 R is a small profinite ring.

From now on we continue the proof in a way which is different from that in [7]. Since we assume that Conjecture 5 holds, there is an open $(A \cup \{a\})$-closed nilpotent (of class n) ideal I of R. Hence there is a natural number $1 \leq k < n$ such that $I^k := \langle I \cdot I \cdots I \rangle$ (k-many times) is infinite but I^{k+1} is finite. Since I^k is compact, there is an open $(A \cup \{a\})$-closed ideal J such that $J \cdot I^k = \{0\}$. Define $H_2 = I^k(a)$. Then H_2 is an infinite $(A \cup \{a\})$-closed group.

The function $f : D \to R$ defined by $f(d)(a) = da$ is continuous. Hence we can find n so that $f[D_n] \subseteq 1 + J$. Take any $h \in H_2$, i.e. $h = i(a)$ for some $i \in I^k$. Then $D_n h = f[D_n](h) \subseteq (1 + J)i(a) = i(a) + Ji(a) = \{i(a)\} = \{h\}$. So we see that D_n fixes H_2 and we are done. □

4. Remarks on Conjecture 5

In this section we shall reduce Conjecture 5 to certain particular cases. First note that we can assume that the characteristic of R is prime, since nilpotency of $R[p] = \{r \in R : pr = 0\}$ and $R/R[p]$ implies nilpotency of R. Second, if R is commutative, we may assume that the nilexponent of R is equal to the characteristic: If $\{r^p : r \in R\} \leq R$ is nilpotent, so is the ideal I generated by it; if R/I is also nilpotent, so is R. This reduces to nilexponent at most p, but rings of nilexponent $< p$ and characteristic p are nilpotent by the Nagata-Higman theorem.

Remark 4.1. *Let R be a profinite nil ring. Then $\bigcap R^n = \{0\}$, where R^n is the ring generated by all products $r_1 \cdots r_n$ for $r_1, \ldots, r_n \in R$.*

Proof: By Fact 2.1 there is a descending sequence $(I_k : k \in \omega)$ of open ideals with trivial intersection. For $k \in \omega$ all rings R/I_k are finite and nil, whence nilpotent of some class n_k. Hence $R^{n_k} \subseteq I_k$, and $\bigcap R^n \subseteq \bigcap R^{n_k} \subseteq \bigcap I_k = \{0\}$. □

Proposition 4.2. *Let R be a small profinite nil ring. If R is not nilpotent, then R^{n+1} is nowhere dense in R^n for all $n > 0$.*

Proof: Suppose for a contradiction that R^{k+1} is open in R^k for some $k > 0$. Then

$$(*) \qquad R^k = (a_0 + R^{k+1}) \cup \cdots \cup (a_m + R^{k+1}),$$

for some $a_0, \dots, a_m \in R^k$. For every $i \leq m$ we can write

$$a_i = \sum_j a_{ij1} a_{ij2} \cdots a_{ijk},$$

for some $a_{ijl} \in R$. Let P be the subring of R generated by all the elements a_{ijl}. Then P is finite by Corollary 2.8.

Claim. *$r_1 \cdots r_n \in P + R^{n+1}$ for all $n \geq k$ and $r_1, \dots, r_n \in R$.*

Proof of Claim: We use induction on n, the case $n = k$ following from the definition of P. So assume tha claim holds for n, and take any $r_1, \dots, r_{n+1} \in R$. By $(*)$ we can find $i \leq m$ such that

$$r_1 \cdots r_{n+1} \in (a_i + R^{k+1})(r_{k+1} \cdots r_{n+1}) \subseteq a_i r_{k+1} \cdots r_{n+1} + R^{n+2}.$$

By inductive hypothesis applied to $a_{ij2}, \dots, a_{ijk}, r_{k+1}, \dots, r_{n+1}$ we get

$$a_{ij1} \cdots a_{ijk} r_{k+1} \cdots r_{n+1} \in a_{ij1}(P + R^{n+1}) \subseteq P + R^{n+2}.$$

Summing over all j yields the claim. □

Thus $R^n \leq P + R^{n+1}$, whence $P + R^n \leq P + R^{n+1}$ for all $n \geq k$, and

$$P + R^k = \bigcap_{n \geq k} (P + R^n) = P + \bigcap_{n \geq k} R^n = P$$

by Remark 4.1. In particular R^k is nilpotent, as is R. □

Let us notice that from Proposition 4.2 we obtain Conjecture 5 for m-stable profinite rings. Of course Theorem 3.3 is much stronger than this, but the above proof is much more elementary than the proof of Theorem 3.3, which uses nontrivial results from [12].

Remark 4.3. *Let R be a small profinite nil ring, and $g_1, \dots, g_n \in R$ be independent generic (i.e. for every $i \leq n$ the orbit $o(g_i/\{g_j : j \neq i\})$ is open in R). If $g_1 \cdots g_n = 0$, then R is nilpotent.*

Proof: Since $g_1 \cdots g_n = 0$, we have $o(g_1/\{g_2, \dots, g_n\}) g_2 \cdots g_n = \{0\}$. Since the orbit $o(g_1/\{g_2, \dots, g_n\})$ is open, there is a $\emptyset$-closed open ideal I_1 such that

$$g_1 + I_1 \subseteq o(g_1/\{g_2, \dots, g_n\}).$$

Hence

$$I_1 g_2 \cdots g_n = \{0\}.$$

Iteratively we obtain a sequence $I_1, \dots, I_n$ of open $\emptyset$-closed ideals such that

$$I_1 \cdots I_n = \{0\}.$$

Now we put $I = I_1 \cap \dots \cap I_n$. Then I is an open nilpotent (of class $\leq n$) ideal in R. Since R/I is finite and nil, whence nilpotent, R itself is nilpotent. □

The next proposition shows that for commutative rings we may assume that the annihilator of any element is rather "small", in particular it is nowhere dense in R.

Proposition 4.4. *To show Conjecture 5 for commutative rings, we can assume that for each $a \in R$ the quotient ring $R/ann(a)$ is not nilpotent.*

Proof: By Theorem 3.1 we may assume that R is small profinite nil of finite nilexponent. Put $I_0 = (0)$, and define inductively I_{n+1} as the set of those $a \in R$ for which $(R/I_n)/ann(a + I_n)$ is nilpotent.

Since $(I_n : n \in \omega)$ is a non-decreasing sequence of $\emptyset$-closed ideals of R, by Fact 2.7 there is a natural number n such that $I_n = I_{n+1}$. Then $(R/I_n)/ann(a + I_n)$ is not nilpotent for all $a \in R \setminus I_n$.

Claim. *For any $k < n$ the ring I_{k+1}/I_k is nilpotent.*

Proof of Claim: Clearly it is enough to show the claim for $k = 0$. Take any generic $g \in I_1$. Then $I_1/ann(g)$ is nilpotent of some class m. Choose any $g_1, \dots, g_m \in I_1$ such that $g, g_1, \dots, g_m$ are independent generic. Then

$$(g_1 + ann(g)) \cdots (g_m + ann(g)) = ann(g).$$

Hence $gg_1 \cdots g_m = 0$, and I_1 is nilpotent by Remark 4.3. □

Thus I_n is nilpotent; this shows the assertion. □

5. Examples

Let F be the free commutative, nilpotent of class 3 ring of characteristic 3 on generators $(x_i : i < \omega)$. For $n < \omega$ let F_n be the ideal generated by $(x_i : i \geq n)$. Then each quotient ring F/F_n is finite nilpotent; their inverse limit is the free, commutative, nilpotent of class 3 profinite ring of characteristic 3 with free topological generators $(y_n : n < \omega)$, where $y_n = (x_n + F_k : k \in \omega)$. We denote this ring by R, and we will use it in several examples below; $Aut^0(R)$ denotes its standard structural group.

Example 1. R is not small.

Proof: Suppose for a contradiction that $(R, Aut^*(R))$ is small for some structural group $Aut^*(R)$. By Fact 2.7 we know that $R^2 := \langle R \cdot R \rangle$ is additively generated by $R \cdot R$ in finitely many steps; say in n steps. Then each $(F/F_\ell)^2$, $\ell \in \omega$, is also additively generated by $F/F_\ell \cdot F/F_\ell$ in n steps.

The ring $(F/F_\ell)^2$ has $3^{\binom{\ell+1}{2}}$ elements, but at most $(3^\ell \cdot 3^\ell)^n = 3^{2\ell n}$ elements of F/F_ℓ are sums of at most n products of two elements, a contradiction for large ℓ. □

The argument easily generalizes to show that no free profinite ring of any nilpotency class > 2 is small.

The next example yields a profinite ring S where S^2 is generated in finitely many steps, but which still is not small.

Example 2. Let I be the topological closure of the ideal of R generated by all the products $y_i y_j$ for $(i,j) \neq (2n, 2n+1)$, $n \in \omega$. Take $S = R/I$ and $y'_n = y_n + I$, $n \in \omega$. Then S is the inverse limit of all quotient rings S/S_n, $n \in \omega$, where S_n is the topological closure of the ideal of S generated by the elements y'_k, $k \geq n$. This system of quotient rings determines a group $Aut^0(S)$. Then S^2 is additively generated by $S \cdot S$ in one step, but $(S, Aut^0(S))$ is not small.

Proof: For the first assertion, note that any element of S^2 is of the form

$$\sum k_{ij} y'_i y'_j = \sum k_n y'_{2n} y'_{2n+1} = \Big(\sum k_n y'_{2n}\Big) \cdot \Big(\sum y'_{2n+1}\Big) \in S \cdot S.$$

Next, suppose for a contradiction that $(S, Aut^0(S))$ is small. For some $n > 0$ we can find a generic element $r \in \langle y'_0, \dots, y'_n \rangle$ such that $r + S_{2n} \subseteq o(r)$. Then $rS_{2n} = \{0\}$, whence $o(r)S_{2n} = \{0\}$ by $\emptyset$-invariance of S_{2n}. But $r + y'_{2n} \in o(r)$ and $y'_{2n+1} \in S_{2n}$, so $y'_{2n} y'_{2n+1} = (r + y'_{2n}) y'_{2n+1} - r y'_{2n+1} = 0$, a contradiction. □

Example 3. Let I be the topological closure of the ideal of R generated by all the products $y_i y_j$ for $i, j \neq 0$. Then I is a $\emptyset$-closed ideal of $(R, Aut^0(R))$. Define S, y'_n, S_n and $Aut^0(S)$ as in Example 2. Then $(S, Aut^0(S))$ is small with an open null ideal, but $ann(S)$ is nowhere dense in S.

Proof: Let T be the topological closure of the subring of S generated by the elements y'_k, $k' \geq 1$. It is easy to see that

$$S = T \oplus y'_0 T \oplus \langle y'_0, y'^2_0 \rangle$$

as an additive group, where $\langle y'_0, y'^2_0 \rangle$ is a subgroup of $(S, +)$ generated by elements y'_0 and y'^2_0. Of course $\langle y'_0, y'^2_0 \rangle \cong {\mathbb{Z}_3}^2$.

Let $Aut^0(T)$ be the standard structural group of T regarded as a group (hence also as a ring, because T is null) with the distinguished basis of open neighbourhoods of identity consisting of the subgroups $T \cap S_n$, $n \geq 1$. Since T has finite characteristic, by [4, Theorem 1.9] we get that $(T, Aut^0(T))$ is small.

On the other hand, the profinite structure $(S, Aut^0(S/\{y'_0, T\}))$ is isomorphic to $(T \times T \times {\mathbb{Z}_3}^2, Aut^0(T))$ with the trivial action of $Aut^0(T)$

on ${\mathbb{Z}_3}^2$. So we get that $(S, Aut^0(S/\{y'_0, T\}))$ is small, hence $(S, Aut^0(S))$ is small, too.

Clearly S_1 is an open null ideal. On the other hand, the ideals S_n, $n \in \omega$, form a basis of open neighbourhoods of 0. But $y'_n \in S_n$ and $y'_0 y'_n \neq 0$. Hence $ann(S)$ does not contain any S_n, and cannot be open in S. So it is nowhere dense. □

Proposition 5.1. *Assume that $(R, Aut^*(R))$ is a small profinite additive group. Expand $(R,+)$ to any ring structure $R^* = (R, +, \cdot)$ so that $Ann(R^*)$ is open in R. Define $Aut^*(R^*)$ as the set of all $f \in Aut^*(R)$ preserving multiplication. Then $(R^*, Aut^*(R^*))$ is a small profinite ring.*

Proof: Since $Ann(R^*)$ is open, we get that the ring multiplication is continuous. Hence we easily get that $(R^*, Aut^*(R^*))$ is a profinite ring. Since $Ann(R^*)$ is open, we can find an open $\emptyset$-closed subgroup G of R which is contained in $Ann(R^*)$.

Take $r_0, \dots, r_n \in R$, representatives of cosets of G. Let S be the subring of R^* generated by $r_0, \dots, r_n$. Since $G \subseteq Ann(R^*)$ has finite index in R^* and R^* has finite characteristic, we get that S is finite.

Fix $g_0, \dots, g_n \in G$. Let A be the group of homeomorphisms of $G \times \{g_1, \dots, g_n\}$ induced by $Aut^*(R/S \cup \{g_1, \dots, g_n\})$. Then $(G \times \{g_1, \dots, g_n\}, A)$ is a small profinite structure.

To finish the proof it is enough to show the following:

Claim. *The profinite structures $(R^*, Aut^*(R^*/S \cup \{g_1, \dots, g_n\})$ and $(G \times \{g_1, \dots, g_n\}, A)$ are isomorphic.*

Proof of Claim: Define $\Phi : G \times \{g_1, \dots, g_n\} \to R$ by

$$\Phi(g, g_i) = g + r_i.$$

We easily see that Φ is a homeomorphism. So it remains to show that the formula

$$\phi(f) = \Phi f \Phi^{-1}$$

gives an isomorphism between groups A and $Aut^*(R^*/S \cup \{g_1, \dots, g_n\})$. The details are left to the reader. □

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