On regular groups and fields

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Abstract

Regular groups and fields are common generalizations of minimal and quasi-minimal groups and fields, so the conjectures that minimal or quasi-minimal fields are algebraically closed have their common generalization to the conjecture that each regular field is algebraically closed. Standard arguments show that a generically stable regular field is algebraically closed. Let K be a regular field which is not generically stable and let p be its global generic type. We observe that if K has a finite extension L of degree n, then $p^{(n)}$ has unbounded orbit under the action of the multiplicative group of L.

Known to be true in the minimal context, it remains wide open whether regular, or even quasi-minimal, groups are abelian. We show that if it is not the case, then there is a counter-example with a unique non-trivial conjugacy class, and we notice that a classical group with one non-trivial conjugacy class is not quasi-minimal, because the centralizers of all elements are uncountable. Then we construct a group of cardinality ω_1 with only one non-trivial conjugacy class and such that the centralizers of all non-trivial elements are countable.

0 Introduction

Recall that a minimal structure is an infinite structure whose all definable subsets are finite or co-finite. A quasi-minimal structure is an uncountable structure in a countable language whose all definable subsets are countable or co-countable. Among the main examples of quasi-minimal structures there are various orders (e.g. $\omega_1 \times \mathbb{Q}$), strongly minimal structures expanded by some orders (see [8, Example 5.1] or Corollary 3.13 in this paper), and Zilber's pseudoexponential fields [11]. A well-known conjecture of Boris Zilber predicts that the complex exponential field $(\mathbb{C}, +, \cdot, 0, 1, \exp)$ is quasi-minimal.

A fundamental theorem of Reineke [9] tells us that each minimal group is abelian. Surprisingly, an analogous statement for quasi-minimal groups seems hard to prove.

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Conjecture 1 *Each quasi-minimal group is abelian.*

The following, more general question was formulated in [8, Section 3] (see Section 1 for the definition of regularity); a variant of this question in a less general context (assuming the existence of a homogeneous pregeometry in a certain sense) was also formulated in [3, Question 1.12].

Question 2 *Is every regular group abelian?*

In Section 3 of this paper, we reduce the problem to the case of groups with only one non-trivial conjugacy class. Then we note that a standard construction (involving HNN-extensions) of an uncountable group with a unique non-trivial conjugacy class does not lead to a quasi-minimal group, because the centralizers of all non-trivial elements of the resulting group are uncountable (and so also co-uncountable). Motivated by this obstacle, we construct a group of cardinality $\aleph_1$ with only one non-trivial conjugacy class, in which the centralizers of all non-trivial elements are countable. We leave as an open question whether the group we constructed is quasi-minimal, or at least regular.

One of the oldest unsolved problems in algebraic model theory is Podewski's conjecture predicting that each minimal field is algebraically closed. Known to be true in positive characteristic [10], it remains wide open in the zero characteristic case. It was recently proved in [6] that it holds when the generic type of the field in question is generically stable (see Section 1 for definitions); in the non generically stable situation, some partial results were obtained (e.g. elimination of the so-called almost linear case).

One has an obvious analog of Podewski's conjecture for quasi-minimal fields.

Conjecture 3 *Each quasi-minimal field is algebraically closed.*

The above conjecture is open even in positive characteristic. A common generalization of Podewski's conjecture and Conjecture 3 is

Conjecture 4 *Each regular field is algebraically closed.*

Applying either the proof of [6, Theorem 1] or of [3, Theorem 1.13], one easily gets

Theorem 5 *Each generically stable regular field is algebraically closed. In particular, each generically stable minimal or quasi-minimal field is algebriacally closed.*

As a consequence, one gets that each quasi-minimal field of cardinality greater than $\aleph_1$ is algebraically closed, and a similar result for regular fields with NSOP. The case of minimal or quasi-minimal fields with NIP is still open.

Having Theorem 5, a natural question arises whether there exists a regular field which is not generically stable. It is conjectured in [6] that there are no such minimal fields, but in the quasi-minimal context such a field exists [8, Example 5.1], and, moreover, it is almost linear using the terminology from [6]. On the other hand, it was proved in [6] that there are no almost linear minimal fields. This reveals a certain difference between minimal and quasi-minimal fields.

In the main result of Section 2, we study the situation when the field in question is not generically stable. So assume now that K (being a monster model) is a regular field which is not generically stable. Suppose that K is not algebrically closed, i.e., it has a finite extension L of degree n. Then L is naturally interpreted as K^n with coordinate-wise addition and some definable multiplication. Let p be the global generic type of K and $p^{(n)}$ its n-th power. We prove that the orbit of $p^{(n)}$ under the multiplicaive group of L is unbounded, which one can hope to be useful to get a final contradiction for some (e.g. NIP) fields. More detailed discussion concerning this appears in Section 2.

The topic of this paper was undertaken in 2009, when Predrag Tanović and the second author made independently various observations and found examples concerning minimal and quasi-minimal groups and fields, in particular the first example of a non generically stable quasi-minimal group (modified later to the context of fields in [8]). Some of these results are contained in [8] and [6], some other ideas were further developed and contained in the first author's Master's thesis completed in June 2011. The current paper is based on this thesis.

The second author would like to thank Clifton Ealy for sharing useful ideas concerning quasi-minimal groups. The observation that a classical group with only one non-trivial conjugacy class is not quasi-minimal was first made by Clifton Ealy.

1 Definitions and basic facts

Throughout this section, $\mathfrak{C}$ denotes a monster model of a first order theory T, i.e., a κ-saturated and κ-strongly homogeneous model for large enough cardinal κ. A global type is a complete type over $\mathfrak{C}$. We say that the set is bounded (or small) if it has cardinality smaller than κ. If not stated otherwise, M denotes a small elementary substructure of $\mathfrak{C}$.

The following notion of strongly regular types in an arbitrary theory has been introduced in [8]; it coincides with the notion of strongly regular types in the stable context.

Definition 1.1 *Let $p(x)$ be a global non-algebraic type. Suppose $\varphi(x) \in p(x)$. We say that $(p(x), \varphi(x))$ is strongly regular (or $p(x)$ is strongly regular via $\varphi(x)$) if, for some small A over which φ is defined, p is A-invariant and for all $B \supseteq A$ and a satisfying $\varphi(x)$, either $a \models p|_B$ or $p|_B \vdash p|_{B,a}$.*

If $(p(x), \varphi(x))$ is strongly regular, then as a witness-set A one can take any set over which p is invariant and φ is defined.

The next fact is a consequence of [8, Proposition 7.1].

Fact 1.2 *Assume that $p(x) \in S(\mathfrak{C})$ is M-definable and non-algebraic. Then $(p(x), \varphi(x))$ is strongly regular if and only if $p|_M \vdash p|_{M\bar{b}}$ for all tuples $\bar{b} \subset \varphi(\mathfrak{C}) \setminus (p|_M)(\mathfrak{C})$.*

One can also define strong regularity (or rather local strong regularity) for types over small models. We give such a definition in the context of definable types.

Definition 1.3 *Assume $p(x) \in S(M)$ is definable and non-algebraic. Let $\varphi(x) \in p(x)$. We say that $(p(x), \varphi(x))$ is strongly regular (or $p(x)$ is strongly regular via $\varphi(x)$) if $p \vdash p|_{M\bar{b}}$ for all tuples $\bar{b} \subset \varphi(\mathfrak{C}) \setminus p(\mathfrak{C})$.*

By Fact 1.2, a definable type $p(x) \in S(M)$ is strongly regular via $\varphi(x) \in p(x)$ in the sense of Definition 1.3 if and only if the unique global heir $\hat{p}(x) \in S(\mathfrak{C})$ of $p(x)$ is strongly regular via $\varphi(x)$ in the sense of Definition 1.1. In particular, we easily get the following remark.

Remark 1.4 *One can apply Definition 1.3 also in the case when $M = \mathfrak{C}$ (taking tuples $\bar{b}$ from a bigger monster model $\mathfrak{C}' \succ \mathfrak{C}$). Then, a definable type $p \in S(\mathfrak{C})$ is strongly regular via $\varphi(x)$ as a global type in the sense of Definition 1.1 if and only if it is strongly regular via $\varphi(x)$ as type over a small model in the sense of Definition 1.3.*

The next fact is an immediate corollary of Definition 1.3 and Fact 1.2 or Remark 1.4.

Fact 1.5 *Let $M \prec \mathfrak{C}$ be small or equal to $\mathfrak{C}$. Assume that $p(x) \in S(M)$ is definable, and let $\bar{p}(x) \in S(M^{eq})$ be the unique type over M^{eq} implied by p. Then $(p(x), \varphi(x))$ is strongly regular in M if and only if $(\bar{p}(x), \varphi(x))$ it is strongly regular in M^{eq}.*

We will also need the following fact whose proof is left as an easy exercise.

Fact 1.6 *Let $M \prec \mathfrak{C}$ be small or equal to $\mathfrak{C}$. Let $p(x) \in S(M)$ be definable and strongly regular via $\varphi(x)$, and assume that f is an M-definable function. If $f(p)$ is non-algebraic, then it is strongly regular via "$x \in f(\varphi(M))$".*

Definition 1.7 *Let p be a global type invariant over A. A Morley sequence in p over A is a sequence $\langle a_\alpha \rangle_\alpha$ such that $a_\alpha \models p|_{Aa_{<\alpha}}$ for all α. The invariant defining scheme for p uniquely determines extensions of p to global types over bigger monster models, so we can talk about a Morley sequence in p over $\mathfrak{C}$ which will be just called a Morley sequence in p. By $p^{(n)}$ we denote $\mathrm{tp}(b_0, \dots, b_{n-1}/\mathfrak{C})$, where $(b_0, \dots, b_{n-1})$ is a Morley sequence in p.*

If p is an A-invariant type, then all Morley sequences of a fixed length in p over A have the same type over A (so the definition of $p^{(n)}$ is correct), and they are indiscernible over A.

Definition 1.8 *A type $p \in S(\mathfrak{C})$ is generically stable if, for a small set A, p is A-invariant and for every formula $\phi(x)$ (with parameters from $\mathfrak{C}$) there exists a natural number N such that for every Morley sequence $(a_i)_{i<\omega}$ in p over A the set $\{i \in \omega : \models \phi(a_i)\}$ or its complement has cardinality less than N.*

Definition 1.9 *Let M be any submodel of $\mathfrak{C}$ (possibly $M = \mathfrak{C}$), and let $p \in S_1(M)$. The operator cl_p is defined on (all) subsets of M by: $cl_p(X) = \{a \in M : a \not\models p|_X\}$. Also, define $cl_p^A(X) = cl_p(X \cup A)$ for any $A \subset M$ and $X \subseteq M$.*

By [8, Lemma 3.1, Corollary 3.1], one has

Fact 1.10 *Let $p \in S_1(\mathfrak{C})$ be a non-algebraic type invariant over A.*
(i) $(p(x), x = x)$ is strongly regular iff cl_p^A is a closure operator on $\mathfrak{C}$.
(ii) Suppose that $(p(x), x = x)$ is strongly regular. Then cl_p^A is a pregeometry operator on $\mathfrak{C}$ iff every Morley sequence in p over A is totally indiscernible over A iff every Morley sequence in p (over $\mathfrak{C}$) is totaly indiscernible over $\mathfrak{C}$ iff p is generically stable.

[8, Theorem 3.1] yields the following dichotomy for a strongly regular type $p \in S_1(\mathfrak{C})$ invariant over A: either p is generically stable, or for some $A_0 \supseteq A$ there is a definable partial order on $\mathfrak{C}$ such that every Morley sequence in p over A_0 is strictly increasing. In particular, we have

Corollary 1.11 *If a strongly regular type is not generically stable, then the theory T has SOP.*

Now, we recall the notion of a regular group.

Definition 1.12 *Let G be a group definable in M. We say that G is regular if there is a definable type $p \in S_G(M)$ which is strongly regular via "$x \in G$" in the sense of Definition 1.3. A field K definable in M is said to be regular if its additive (equivalently multiplicative) group is regular.*

Recall that by our convention $M \prec \mathfrak{C}$ is small. But any model of T is small in some monster model, so the above definition can be applied to groups definable in any model M of T, in particular to groups definable in $\mathfrak{C}$. Namely, a group G definable in $\mathfrak{C}$ is regular if there is a definable type $p \in S_G(\mathfrak{C})$ which is strongly regular via "$x \in G$" in the sense of Definition 1.3, or, equivalently, in the sense of Definition 1.1 by Remark 1.4.

We should recall here that in [8] regular groups were defined as those groups G definable in the monster model $\mathfrak{C}$ for which there exists a type $p \in S_G(\mathfrak{C})$ which is strongly regular via "$x \in G$" in the sense of Definition 1.1. However, by [8, Theorem 3.2], we know that such a type p is always definable over the parameters over which G is defined, which together with Fact 1.2 and Remark 1.4 yields the following remark, justifying our definition of regular groups.

Remark 1.13 *Let G be a group definable in a small model $M \prec \mathfrak{C}$ or in $M = \mathfrak{C}$. Let G^* be the interpretation of G in the monster model $\mathfrak{C}$. Then, G is regular if and only if there exists a type $p \in S_{G^*}(\mathfrak{C})$ which is strongly regular via "$x \in G^*$".*

The following fact is [8, Theorem 3.2].

Fact 1.14 *Suppose that G is a group definable over $\emptyset$ in $\mathfrak{C}$, which is regular, witnessed by $p(x) \in S_G(\mathfrak{C})$. Then:*
(i) $p(x)$ is both left and right translation invariant (and, in fact, invariant under definable bijections).
(ii) a formula $\varphi(x)$ is in $p(x)$ iff two left [right] translates of $\varphi(x)$ cover G iff finitely many left [right] translates of $\varphi(x)$ cover G. Hence, $p(x)$ is a unique (global) generic (in the sense of translates) type of G.
(iii) $p(x)$ is definable over $\emptyset$.
(iv) $G = G^0$ (i.e., G is connected).

We say that an element a from a regular group G is generic over A if $a \models p|_A$, where p is the unique global generic type of G.

Corollary 1.15 *Let K be a regular field with prime field F. If $a \in K$ is a generic over $A \subset K$, elements c, d belong to $F(A)$ and $c \neq 0$, then $ac + d$ is generic over A.*

We will need the following corollary of Facts 1.5 and 1.6.

Corollary 1.16 *Let G be a regular group definable in a model M, and let H be its definable subgroup of infinite index. Then G/H is also regular, as a group definable in M^{eq} (after addding to the language parameters over which G and H are defined).*

Proof. By Fact 1.5, G is regular in M^{eq}. Let $p = \mathrm{tp}(g/M^{eq})$ be the unique generic type of G over M^{eq}. Let H^* be the interpretation of H in a monster model extending M. Note that the imaginary element gH^* is not algebraic over M^{eq}, as otherwise the generic element g belongs to an M-definable union of finitely many cosets of H^*, and so H is of finite index in G, a contradiction. Taking in Fact 1.6 the surjection $f : G \to G/H$ given by $f(x) = xH$, we conclude that $\mathrm{tp}(gH^*/M^{eq})$ is strongly regular via "$y \in G/H$", so G/H is regular. $\square$

Definition 1.17 *A regular group is said to be generically stable, if its unique global generic type is generically stable (over $\emptyset$), and similarly for fields.*

Talking about regularity of a group or field G definable in a model M, one should focus on the sort "$x \in G$". However, the next remark tells us that for aesthetic reasons one can assume that $M = G$ (possibly with extra relations).

Remark 1.18 *(i) If a group G definable in a model M is regular, then G treated as a group definable in G equipped with the structure induced from M is also regular.*
(ii) If a regular group G definable in a model M is generically stable, then G treated as a group definable in G equipped with the structure induced from M is also generically stable.

Assume M is a minimal [or quasi-minimal] structure. Let $p \in S_1(M)$ be the type consisting of all formulas over M which define co-finite [co-countable, respectively] subsets of M. Not accidentally, this type is called the generic type of M. Namely, assume now that $(M, \cdot, \dots)$ is a minimal or quasi-minimal group and $\varphi(x)$ is any formula; then $\varphi(x) \in p$ iff $\varphi(M) \cdot \varphi(M) = M$ iff two left [right] translates of $\varphi(M)$ cover M iff finitely many left [right] translates of $\varphi(M)$ cover M. In particular, p is definable over $\emptyset$, and so it has a unique global heir $\hat{p}$ in $S(\mathfrak{C})$ (where $\mathfrak{C} \succ M$ is a monster model). It is not hard to check that cl_p is a closure operator on M, so $cl_{\hat{p}}$ is a closure operator on $\mathfrak{C}$ by the definability of p, and thus $(\hat{p}, x = x)$ is strongly regular (see [8, Theorem 5.3]).

Corollary 1.19 *A minimal or quasi-minimal group is regular.*

Finally, we recall fundamental issues concerning HNN-extensions, which will be used in the last section. For more details the reader is referred to [7, Chapter IV].

Fact 1.20 *If G is a group, $A, B \leq G$ and $\varphi : A \to B$ is an isomorphism, then the group F defined by the presentation*

$$\langle G, t | \forall a \in A(tat^{-1} = \varphi(a)) \rangle$$

contains G as a subgroup, and it is called the HNN-extension of G relative to A, B and φ. Moreover, every element of F of finite order is conjugated with some element of G (also of finite order).

Notice that the free product $G * \mathbb{Z}$ can be treated as an HNN-extension of G by putting $A = B = \{e\}$ and $\varphi = \mathrm{id}$.

Definition 1.21 *With the previous notation, we say that a word $w = g_0 \dots g_n$, which represents an element of F, is reduced if it contains no subwords: tt^{-1}; $t^{-1}t$; tat^{-1} for an element $a \in A$; $t^{-1}bt$ for $b \in B$; gh for $g, h \in G$; e, unless $w = e$.*

A word is cyclically reduced if every cyclic permutation of it is reduced. An element of F is cyclically reduced if it has a representation as a cyclically reduced word.

Definition 1.22 *With the previous notation, let $S_A \ni e$ and $S_B \ni e$ be transversals of right cosets of the groups A and B in G . We say that a reduced word $w = g_0 \dots g_n$ is in normal form if for every subword of w of the form $t^{-1}g$ for $g \in G$ we have that $g \in S_B \setminus \{e\}$, and for every subword of the form tg for $g \in G$ we have that $g \in S_A \setminus \{e\}$.*

We will sometimes write reduced words as $a_0 t^{\epsilon_1} a_1 t^{\epsilon_2} a_2 \dots t^{\epsilon_m} a_m$, where $\epsilon_i \in \{-1, 1\}$ and $a_i \in G$. Since our definition of reduced words do not allow neutral elements as letters, whenever some of the a_i's in the above word are neutral, this word is treated as the word in which all these neutral a_i's have been removed.

The following, well-known fact is called “The Normal Form Theorem for HNN-Extensions”.

Fact 1.23 *With the previous notation:*
(i) every element of F has a representation as a reduced word;
(ii) every element of F has a unique representation as a word in normal form; in order to write an element from F in normal form, it is is enough to apply an appropriate sequence of relations of the form $tat^{-1} = \varphi(a)$ and $t^{-1}\varphi(a)t = a$ with $a \in A$, starting from the end of the word and moving to the left;
(iii) a non-trivial (i.e., different from e) reduced word represents a non-trivial element of F;
(iv) if $u = a_0t^{\epsilon_1} \dots t^{\epsilon_m}a_m$ and $v = b_0t^{\delta_1} \dots t^{\delta_n}b_n$ are reduced words, then $m = n$ and $\epsilon_i = \delta_i$ for $i = 1, \dots, n$; thus, one can define $|u|_t$ as the number of letters $t^{\pm 1}$ in some (equivalently any) reduced word representing u.

Lemma 1.24 *If $a, b \in G * \mathbb{Z}$, $e \neq ab = (ba)^{-1}$ and G does not contain an involution, then $ab \in G^{G*\mathbb{Z}}$.*

Proof. Since $G^{G*\mathbb{Z}}$ is closed under conjugation, we can conjugate a and b by an arbitrary (but the same) element of $G * \mathbb{Z}$. Hence, as any element of the free product is conjugated with a cyclically reduced one, wlog b is cyclically reduced. The above equality is equivalent to $b^2 = (a^{-1})^2$. If a is cyclically reduced, then $a \in G$ (otherwise there are no reductions after concatenation of the words a^{-1} and a^{-1}, and so, comparing letters, we get $b = a^{-1}$, a contradiction), therefore also $b \in G$, and we are done. If a is not cyclically reduced, then, since a^{-2} is cyclically reduced (as b^2 is such), a^{-2} has to be equal to e. It is a contradiction, because, by Fact 1.20, $G * \mathbb{Z}$ does not contain an involution. □

2 Some properties of regular fields

By Remarks 1.13 and 1.18, when we talk about a regular field K in our results below, without loss of generality we assume that $K = \mathfrak{C}$ (possibly with extra relations) is a monster model; but when K is quasi-minimal, it is, of course, not necessarily a monster model; the formulation of Theorem 2.4 makes sense only working in a monster model. We start from the following basic observation.

Proposition 2.1 *Let K be a regular field. Then the function $x \mapsto x^n$ is onto K for every natural number $n > 0$, and also the function $x \mapsto x^q - x$ is onto K if $q > 0$ is the characteristic of K. In particular, K is radically closed.*

Proof. Let p be the global generic type of K and $g \models p|_\emptyset$. Let $f \colon K \to K$ be one of the functions considered in the proposition. Then g is algebraic over $f(g)$, so $p|_\emptyset$ cannot imply $p|_{f(g)}$ (as p is non-algebraic). Hence, by the definition of regularity, $f(g) \models p|_\emptyset$, and so $f[K]$ is generic. Since this is a multiplicative or an additive subgroup, and regular groups are connected, we get that $f[K] = K$. □

Let K be a regular field, and let $p \in S_1(K)$ be its global generic type. If K is generically stable, then (K, cl_p) is an infinite dimensional pregeometry with the homogeneity property that for any finite $A \subset K$, $K \setminus cl_p(A)$ forms an orbit under $Aut(K/A)$. Therefore, [3, Theorem 1.13] or an obvious adaptation of the proof of [6, Theorem 1] yields

Theorem 5 *Each generically stable regular field is algebraically closed. In particular, each generically stable minimal or quasi-minimal field is algebriacally closed.*

Theorem 5 together with Corollary 1.11 give us the following conclusion.

Corollary 2.2 *If the theory of a regular field K is simple (or just has NSOP), then K is generically stable, so algebraically closed.*

The next corollary follows from Theorem 5 and [8, Corollary 5.1] which says that the global generic type of a quasi-minimal structure of cardinality greater than $\aleph_1$ is generically stable. Working in the context of groups, we observed [8, Corollary 5.1] independently in 2009, and we give here our short proof (not using the dichotomy theorem). Recall that a corresponding, well-known fact for minimal structures says that each uncountable minimal structure is strongly minimal.

Corollary 2.3 *If K is a quasi-minimal field and $|K| > \aleph_1$, then K is generically stable, so algebraically closed.*

Proof. Let $p \in S_1(K)$ be the generic type of K. Since over any set of parameters A of cardinality $\leq \aleph_1$ there are at most $\aleph_1$ countable definable sets, $cl_p(A) \neq K$. Hence, we can choose in K a Morley sequence $(b_\alpha)_{\alpha<\omega_2}$ in $\hat{p}$ over $\emptyset$ (where $\hat{p}$ is the unique global heir of p). In order to show that this Morley sequence is totally indiscernible, it is enough to show that for any fixed $n \in \omega$ the type of the sequence $(b_0, \dots, b_n)$ is invariant under transpositions of adjacent elements. Since we can treat a prefix of the Morley sequence as parameters, we can focus on the transposition $(1, 2)$. Consider any formula ϕ such that $\models \phi(b_1, \dots, b_n)$. By the indiscernibility of the sequence $(b_\alpha)_{\alpha<\omega_2}$, we have that $\phi(x, b_{\omega_1 2}, \dots, b_{\omega_1 n})$ is satisfied by all elements $b_{<\omega_1}$. Thus, quasi-minimality implies that it is satisfied by co-countably many elements, in particular by the element b_α for some $\omega_1 2 < \alpha < \omega_1 3$. Since $\models \phi(b_\alpha, b_{\omega_1 2}, \dots, b_{\omega_1 n})$, the indiscernibility of the sequence $(b_\alpha)_{\alpha<\omega_2}$ gives us $\models \phi(b_2, b_1, b_3, \dots, b_n)$. □

The above results lead to a question whether every quasi-minimal [regular] field is generically stable. Such a question for groups was formulated by Pillay. Tanović and independently the second author found a counter-example, which was later modified to the context of fields. An example of a quasi-minimal, non generically stable field can be found in [8, Example 5.1]; it is an algebraically closed field with some ordering. A classification of minimal almost linear (so non generically stable) groups was found in [6]. At the end of this paper, we give the second author's original example of a non generically stable minimal group; an interesting property is that it has an elementary extension which is quasi-minimal.

These examples show that Theorem 5 is too weak to deduce Conjectures 3 or 4 in their full generality (it was strong enough in the last two corollaries, because we used some extra assumptions). An interesting question is what happens if we additionally assume NIP. The example of a non-generically stable quasi-minimal group given at the end of the paper has NIP which suggests that Theorem 5 may be too weak even in the NIP context; but we do not know whether the only known example of a non generically stable field [8, Example 5.1] has NIP.

Below we prove a theorem about regular fields which may have applications in proofs of algebraic closedness of fields that satisfy some additional assumptions (like NIP), which we discuss after the proof.

Theorem 2.4 *Let K be a regular field with the global generic type p. Assume that K has a proper, finite extension L of degree n (so L can be identified with K^n with the coordinate-wise addition and definable multiplication). Then $p^{(n)}$ has unbounded orbit under the action of the multiplicative group of L.*

Proof. Suppose for a contradiction that $p^{(n)}$ has bounded orbit. We will show that K is generically stable, so, by Theorem 5, it is algebraically closed, which yields a contradiction with the assumption that L was a proper, algebraic extension.

By Proposition 2.1, K is perfect. So $L = K(a)$ for some a, and let

$$f = X^n - b_{n-1}X^{n-1} - \ldots - b_0$$

be the minimal polynomial for a over K. By the boundedness of the orbit, there exists an uncountable set $X \subseteq K$ such that for every $\alpha, \beta \in X$ one has

$$(\beta a + 1)p^{(n)} = (\alpha a + 1)p^{(n)}.$$

We take two elements α and β from X which are algebraically independent over the coefficients of f. We have

$$(\alpha a + 1)^{-1}(\beta a + 1)p^{(n)} = p^{(n)}.$$

For each element $a' \in L$ the function $x \mapsto a' \cdot x$ is a linear map from L into L treated as a vector space over K. Choose $(1, a, a^2, \ldots, a^{n-1})$ as a basis of the vector space L over K; then the elements considered above can be viewed as the following matrices (put $m = n - 1$):

$$\alpha a + 1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & \alpha b_0 \\ \alpha & 1 & 0 & 0 & \ldots & \alpha b_1 \\ 0 & \alpha & 1 & 0 & \ldots & \alpha b_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 + \alpha b_m \end{pmatrix} \qquad \beta a + 1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots & \beta b_0 \\ \beta & 1 & 0 & 0 & \ldots & \beta b_1 \\ 0 & \beta & 1 & 0 & \ldots & \beta b_2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 1 + \beta b_m \end{pmatrix}.$$

Let $q_j = \sum_{i=0}^{j}(-\alpha)^{j-i}b_i$ and $h = \frac{1}{1+\alpha q_m}$. Note that since $b_0 \neq 0$, the algebraic independence of α implies that $q_0, \ldots, q_m \neq 0$ and also $1 + \alpha q_m \neq 0$, and so h is well-defined. Then $(\alpha a+1)^{-1}$ equals

$$\begin{pmatrix} 1+(-\alpha)^{m+1}q_0h & (-\alpha)^m q_0 h & (-\alpha)^{m-1}q_0h & \dots & (-\alpha)q_0h \\ -\alpha+(-\alpha)^{m+1}q_1h & 1+(-\alpha)^m q_1 h & (-\alpha)^{m-1}q_1h & \dots & (-\alpha)q_1h \\ (-\alpha)^2+(-\alpha)^{m+1}q_2h & -\alpha+(-\alpha)^m q_2 h & 1+(-\alpha)^{m-1}q_2h & \dots & (-\alpha)q_2h \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-\alpha)^m h & (-\alpha)^{m-1}h & (-\alpha)^{m-2}h & \dots & h \end{pmatrix}.$$

Let $M=(\alpha a+1)^{-1}(\beta a+1)$. One can show that the algebraic independence of α and β implies that the entries of M are non-zero. We will only need to know that

$$M_{m-1,m}=\sum_{i=0}^{m-1}\beta b_i((-\alpha)^{(m-1-i)}+(-\alpha)^{(m+1-i)}q_{m-1}h)-(1+\beta b_m)(\alpha q_{m-1}h)\neq 0,$$

which follows from the fact that $\alpha q_{m-1}h\neq 0$ and the algebraic independence of α and β over the field generated by the coefficients of f.

We choose a Morley sequence $(c_0,\dots,c_m)$ in p over an arbitrary, but fixed, small set of parameters B, containing the entries of M. Using the fact that $M\cdot p^{(n)}=p^{(n)}$ and Corollary 1.15, we get

$$\mathrm{tp}(c_{m-1},c_m/B)=\mathrm{tp}((Mc)_{m-1},(Mc)_m/B)=\mathrm{tp}(\textstyle\sum_{i=0}^m M_{m-1,i}c_i,\sum_{i=0}^m M_{m,i}c_i/B)= \mathrm{tp}(\textstyle\sum_{i=0}^m M_{m-1,i}c_i,\sum_{i=0}^{m-1}(M_{m,i}-\frac{M_{m-1,i}M_{m,m}}{M_{m-1,m}})c_i/B).$$

Let $d=\sum_{i=0}^m M_{m-1,i}c_i$ and $e=\sum_{i=0}^{m-1}(M_{m,i}-\frac{M_{m-1,i}M_{m,m}}{M_{m-1,m}})c_i$. Since c_m is generic over $c_{<m}\cup B$ and $M_{m-1,m}\neq 0$, by Corollary 1.15, we have that d is generic over $c_{<m}\cup B$, so also over $\{e\}\cup B$. We conclude that

$$\mathrm{tp}(c_{m-1},c_m/B)=\mathrm{tp}(d,e/B)=\mathrm{tp}(c_m,c_{m-1}/B).$$

We have proved that each 2-element Morley sequence in p over any small set of parameters B (containing the entries of M) is totally indiscernible over B (i.e., its type over B is invariant under the transposition). This implies that p is generically stable. □

As it was mentioned, we hope that Theorem 2.4 may turn out to be useful in proving that some regular fields are algebraically closed. The idea is of course to assume that our regular field K has a proper, finite extension L of degree n and prove somehow that the orbit $L\cdot p^{(n)}$ must be bounded (where p is the global generic type of K).

Let us look at this in the NIP context. A (global) Keisler measure is a finitely additive, probabilistic measure μ on definable subsets of a given monster model $\mathfrak{C}$. A type q is μ-generic if for every $\varphi(x)\in q$, $\mu(\varphi(\mathfrak{C}))>0$. From [2], it follows that in theories with NIP, there are boundedly many global μ-generic types. Assume K is a regular field satisfying NIP and having a proper, finite extension L of degree n. If we could find a Keisler measure on L invariant under the non-zero multiplicative translations (or satisfying a weaker condition that the action of the multiplicative group of L moves μ-positive sets onto μ-positive sets) and such that $p^{(n)}$ is μ-generic, then the type $p^{(n)}$ would have bounded orbit, and we would be done. It is easy to show (see [4, Section 4]) that there exists a (definable) Keisler measure on L invariant under the additive and non-zero multiplicative translations, but it is not clear how to construct such a measure μ so that $p^{(n)}$ is μ-generic.

Let us finish with a comment on positive characteristic. The proof of Podewski's conjecture in positive characteristic [10] can be divided into two steps. First, using the Frobenius automorphism and the fact that minimal fields are radically closed, one gets that, working in the pure field structure, $\mathrm{acl}(\emptyset) = F^{\mathrm{alg}}$ (i.e., the model-theoretic algebraic closure of $\emptyset$ coincides with the field-theoretic algebraic closure of the prime field), and this part goes through in the quasi-minimal context. Next, using Tarski-Vaught test, one checks that $\mathrm{acl}(\emptyset)$ is an elementary substructure of the field in question, which completes the proof. However, this part does not go through for quasi-minimal fields, as a co-countable set need not to intersect the countable field $\mathrm{acl}(\emptyset) = F^{\mathrm{alg}}$.

3 Examples around quasi-minimal groups

Our first result shows that if there exists a counter-example to Conjecture 1 or Question 2, then there is a counter-example with only one non-trivial conjugacy class.

Lemma 3.1 *If a group G is regular [or quasi-minimal] and non-abelian, then $G/Z(G)$ is also regular [resp. quasi-minimal] and it has a single non-trivial conjugacy class (so it is non-abelian and torsion-free).*

Proof. Let us consider the case when G is regular. Since, by Fact 1.14, G is connected (i.e., with no definable subgroups of finite index) and $Z(G)$ is definable and proper, regularity of $G/Z(G)$ follows from Corollary 1.16. The fact that $G/Z(G)$ has a unique non-trivial conjugacy class was proved in [5, Proposition 3.7]. For the reader's convenience we repeat this proof simplifying it slightly.

By Remark 1.18, for simplicity we can assume that the model M in which G is defined is just G (with some extra relations). Consider any $a \in G \setminus Z(G)$. Take $g \models p|_G$, where p is the global generic type of G.

Notice that if $a^g \models p|_G$, then the formula defining the conjugacy class of a belongs to p. Thus, all elements $a \in G \setminus Z(G)$ for which $a^g \models p|_G$ are in one conjugacy class. So, it remains to show that the assumption $a^g \not\models p|_G$ leads to contradiction.

This assumption and the strong regularity of p via "$x = x$" imply that $tp(g/G) \vdash tp(g/a^g, G)$. Thus, there is a formula $\varphi(x, y)$ (without parameters) and $b \in G^n$ such that $g \models \varphi(x, b)$ and $\models (\varphi(x, b) \rightarrow a^x = a^g)$. So, there is $c \in G$ such that $a^c = a^g$, and hence $g \in cC(a)$. This means that $p \vdash$ "$x \in cC(a)$". But $C(a)$ is a proper definable subgroup of G, so it has infinite index in G, hence $cC(a)$ is non-generic, a contradiction.

By virtue of Corollary 1.19, the quasi-minimal case follows from what we have just proved in the regular case and an easy observation that if G is quasi-minimal, then $G/Z(G)$ is also quasi-minimal.

To finish the proof of the lemma, let us recall a standard argument showing that each non-abelian group with only one non-trivial conjugacy class is torsion-free. Of course, all non-trivial elements have the same order which is infinite or equal to some prime number p. Assume that the second possibility holds. Clearly $p \neq 2$, as otherwise G is abelian. Let $x \in G \backslash \{e\}$. There is y such that $yxy^{-1} = x^2$. Then, by Fermat's little theorem, we have

$$x = y^p x y^{-p} = x^{2^p} = \left(x^{2^{p-1}}\right)^2 = x^2.$$

Hence, $x = e$, a contradiction. □

By virtue of Lemma 3.1, a natural approach to find a counter-example to Conjecture 1 is to analyze methods of constructing groups with only one non-trivial conjugacy class. A fundamental tool in such constructions are HNN-extensions. A classical construction goes as follows:

$$\begin{array}{l} G_0 = F_{\aleph_1}, \\ G_{i+1} = \langle G_i \cup \{T_{st} : s,t \in G_i \backslash \{e\}\} | T_{st} s T_{st}^{-1} = t \rangle, \\ G = \bigcup_{i \in \omega} G_i. \end{array} \qquad (1)$$

Remark 3.2 *The group G constructed above is uncountable, it has one non-trivial conjugacy class, but it is not quasi-minimal as the centralizers of all non-trivial elements are uncountable.*

Proof: By a basic property of HNN-extensions, we know that $G_0 < G$, so G is uncountable. Every two elements of G have to belong to G_i for some i, so they are conjugated in G_{i+1}.

Let $r, s, t \in G_0 \backslash \{e\}$. Then $T_{st} T_{rs} T_{tr} \in C_{G_1}(t)$. Since G_0 is uncountable, there exist uncountably many pairs $(r, s) \in G_0 \times G_0$. Moreover, for distinct pairs $(r_1, s_1) \neq (r_2, s_2)$ the elements $T_{s_1 t} T_{r_1 s_1} T_{t r_1}$ and $T_{s_2 t} T_{r_2 s_2} T_{t r_2}$ are also distinct. Therefore, $C_G(t)$ is uncountable. Hence, G is not quasi-minimal. □

We do not know whether the above group is regular. A possible way to prove that it is not regular could be to show that there is no global generic type, or that there are more than one global generic types. Notice, however, that this group is not stable, as there are no stable groups with a unique non-trivial conjugacy class.

A reason why the above construction produces too big centralizers (and, in consequence, the group is not quasi-minimal) are "redundant conjugations". We modify this construction below, producing an uncountable group with a unique non-trivial conjugacy class and such that all non-trivial elements have countable centralizers, but we leave as an open question whether this group is quasi-minimal or at least regular.

Let G_0 be a countable group with one non-trivial conjugacy class (such a group can be constructed by starting in the above construction from a countable free group). From now on, we fix an element $x \in G_0 \backslash \{e\}$. For $H \geq G_0$, S_H denotes the conjugacy class x^H. By recursion, we construct a sequence of countable, torsion-free groups $(G_\alpha)_{\alpha<\omega_1}$ along with functions $f_\alpha : G_\alpha \setminus (S_{G_\alpha} \cup \{e\}) \to G_\alpha$, which will satisfy:

(i) $G_\alpha \lneq G_{\alpha+1}$,

(ii) $\bigcup_{\alpha<\omega_1} G_\alpha = \bigcup_{\alpha<\omega_1} S_{G_\alpha}$,

(iii) if $y \in S_{G_\alpha}$, then $C_{G_\alpha}(y) = C_{G_{\alpha+1}}(y)$,

(iv) if $y \in G_{\alpha_1} \setminus (S_{G_{\alpha_1}} \cup \{e\})$ and $y \in G_{\alpha_2} \setminus (S_{G_{\alpha_2}} \cup \{e\})$, then $f_{\alpha_1}(y) = f_{\alpha_2}(y)$; in other words, for a given y the value $f_\alpha(y)$ does not depend on the choice of α such that $y \in G_\alpha \setminus S_{G_\alpha}$, and this common value $f_\alpha(y)$ will be denoted by z_y,

(v) if $e \neq y \in G_\alpha \backslash S_{G_\alpha}$, then $\exists n \in \mathbb{N}(z_y^n = y \wedge \forall w \in G_\alpha \forall m \in \mathbb{N}\backslash\{0\}(wy^m w^{-1} \in \langle z_y \rangle \Rightarrow w \in \langle z_y \rangle))$.

Let $G = \bigcup_{\alpha<\omega_1} G_\alpha$ and $Z_\alpha = \{z_y : y \in G_\alpha \backslash (S_{G_\alpha} \cup \{e\})\}$. Condition (i) implies that G is uncountable, (ii) that there exists only one non-trivial conjugacy class, and (iii) that the centralizers of non-trivial elements are countable. The last two conditions will allow to maintain the others during the construction.

Suppose $(G_\alpha)_{\alpha<\beta}$ and $(f_\alpha)_{\alpha<\beta}$ have been constructed so that Conditions (i) and (iii) hold for all $\alpha < \beta$ for which $\alpha + 1 < \beta$, Condition (iv) holds for all $\alpha_1, \alpha_2 < \beta$, Condition (v) holds for all $\alpha < \beta$, and all G_α's are countable and torsion-free. We describe how to construct G_β and f_β.

Case 1 (limit step) $\beta \in \mathrm{LIM}$.
We put $G_\beta := \bigcup_{\alpha<\beta} G_\alpha$. Conditions (i) and (iii) are clearly satisfied, since they were satisfied for every $\alpha < \beta$; G_β is clearly countable and torsion-free. Any $y \in G_\beta \setminus S_{G_\beta}$ belongs to $G_\alpha \setminus S_{G_\alpha}$ for some $\alpha < \beta$, so we can put $f_\beta(y) := f_\alpha(y) = z_y$ and then Condition (iv) is automatically satisfied for $\alpha_1, \alpha_2 \leq \beta$. Since z_y's satisfy Condition (v) for all $\alpha < \beta$, we easily see that (v) holds for β, too.

Case 2 (odd step) $\beta = \alpha + 1 = \delta + 2n + 1$ for some $\delta \in \mathrm{LIM} \cup \{0\}$ and $n \in \mathbb{N}$.
We define

$$G_{\alpha+1} := G_\alpha * \mathbb{Z},$$

and for $y \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$ we put $f_\beta(y) := f_\alpha(y) = z_y$ (the definition of $f_\beta(y)$ for other arguments $y \in G_{\alpha+1} \setminus (S_{G_{\alpha+1}} \cup \{e\})$ is given in the proof of (v) below), and we will check that our conditions hold.

By Fact 1.20 and the assumption that G_α is countable and torsion-free, we get that $G_{\alpha+1}$ is countable and torsion-free.

(i) is clearly satisfied.

(iii) Let $y \in S_{G_\alpha}$. Suppose for a contradiction that there is some element $k \in C_{G_{\alpha+1}}(y) \backslash C_{G_\alpha}(y)$ and let $a_0 t^{\epsilon_1} a_1 \dots t^{\epsilon_m} a_m$ be its normal form. Then we have that $a_0 t^{\epsilon_1} a_1 \dots t^{\epsilon_m} a_m y a_m^{-1} t^{-\epsilon_m} \dots t^{-\epsilon_1} a_0^{-1} = y$. If $a_m y a_m^{-1} \neq e$, then the term on the left defines an element from outside G_α (because $m \geq 1$ since k is outside G_α), a contradiction. The equality $a_m y a_m^{-1} = e$ implies that $y = e$, which is also a contradiction.

(iv) Regardless of the full definition of f_β, the partial definition of f_β given above guarantees that (iv) holds for $\alpha_1, \alpha_2 \leq \beta$.

(v) Let $y \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$. We will check that since z_y satisfies Condition (v) for y and α, it also satisfies it for y and $\alpha + 1$. It is enough to show that for every $k \in G_{\alpha+1} \backslash G_\alpha$ and $m \in \mathbb{N} \setminus \{0\}$ we have $k y^m k^{-1} \notin \langle z_y \rangle$. Arguing as in (iii), we get that if $k y^m k^{-1} \in \langle z_y \rangle \subset G_\alpha$, then $y^m = e$, and so $y = e$, which contradicts the choice of y.

Let $y \in G_\alpha^{G_{\alpha+1}} \setminus (S_{G_{\alpha+1}} \cup G_\alpha)$. Then $y = y_\alpha^s$ for some $y_\alpha \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$ and $s \in G_{\alpha+1}$. Define $f_\beta(y) = z_y := z_{y_\alpha}^s$. It follows from the above paragraph and the fact that conjugation is a group automorphism that (v) holds for y and $\alpha + 1$.

Let $y \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$. Since conjugation is a group automorphism, we can assume that y is represented as a cyclically reduced word $t^{\epsilon_1} a_1 \dots t^{\epsilon_s} a_s$. Define $f_\beta(y) = z_y$ as the shortest period of the word $t^{\epsilon_1} a_1 \dots t^{\epsilon_s} a_s$; in particular, we get $z_y^j = y$ for some $j \in \mathbb{N} \setminus \{0\}$. Let k be such that $k y^m k^{-1} = z_y^l$ for some $m \in \mathbb{N} \setminus \{0\}$ and $l \in \mathbb{Z}$. We choose $r \in \mathbb{N}$ so that $|k|, |l| \ll r$. By comparing the numbers of letters $t^{\pm 1}$ in the equation $k y^{mr} k^{-1} = z_y^{lr}$ (notice that z_y is cyclically reduced, so either k fully cancels with a prefix of y^{mr} and there are no cancellations between y^{mr} and k^{-1}, or the other way around), we get that $l = \pm mj$. In both cases below, we consider a long enough prefix or suffix of both sides of the equality $k y^{rm} k^{-1} = z_y^{lr}$, depending on where there are no cancellations. If $l = -mj$, there would exist a division $z_y = ab$ such that $ba = z_y^{-1}$. By Lemma 1.24, we would get $z_y \in G_\alpha^{G_{\alpha+1}}$, so $y \in G_\alpha^{G_{\alpha+1}}$, a contradiction. So it must be the case that $l = mj$. From the equation $k y^{rm} k^{-1} = z_y^{lr}$, we conclude that either $k = z_y^i$ for some $i \in \mathbb{Z}$ and we are done, or there exists a proper division $z_y = ab$ such that $z_y = ba$, which implies that z_y is periodic, a contradiction with the choice of z_y.

Case 3 (even step) $\beta = \alpha + 1 = \delta + 2n$ for some $\delta \in \mathrm{LIM} \cup \{0\}$ and $n \in \mathbb{N} \setminus \{0\}$. We define

$$G_{\alpha+1} := \langle G_\alpha, t | t x t^{-1} = z \rangle$$

for some $z \in Z_\alpha$ (if $Z_\alpha = \emptyset$, i.e., $S_{G_\alpha} = G_\alpha \setminus \{e\}$, then we could define G_α as in Case 2; but, in fact, Z_α is non-empty after every "odd" step). At the end of the construction, we will describe how to choose the elements z at these "even" steps of the construction in order to satisfy Condition (ii) after the whole construction. Now, we prove the other conditions. For $y \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$ we put $f_\beta(y) := f_\alpha(y) = z_y$

(the definition of $f_\beta(y)$ for other arguments $y \in G_{\alpha+1} \setminus (S_{G_{\alpha+1}} \cup \{e\})$ is given in the proof of (v) below)

As in Case 2, by Fact 1.20 and the assumption that G_α is countable and torsion-free, we get that $G_{\alpha+1}$ is countable and torsion-free.

(i) obviously holds.

(iii) Since all elements from S_{G_α} are conjugated by elements from G_α, it is enough to check (iii) only for x. Suppose for a contradiction that there exists an element $k \in C_{G_{\alpha+1}}(x) \setminus C_{G_\alpha}(x)$, and write it in reduced form as $a_0 t^{\epsilon_1} a_1 \dots t^{\epsilon_m} a_m$. Then $a_0 t^{\epsilon_1} \dots t^{\epsilon_m} a_m x a_m^{-1} t^{-\epsilon_m} \dots t^{-\epsilon_1} a_0^{-1} = x$. If $\epsilon_1 = 1$, then x would be conjugated in G_α with some integer power z^s for $s \neq 0$. Since x is conjugated with x^2 in G_α, z^s would be conjugated with z^{2s}, which violates Condition (v) for G_α. (Indeed, suppose $bz^s b^{-1} = z^{2s}$ for some $b \in G_\alpha$. We know that $z = z_y$ for some $y \in G_\alpha \setminus (S_{G_\alpha} \cup \{e\})$. By (v), there is n such that $z_y^n = y$. So $by^s b^{-1} = y^{2s} \in \langle z_y \rangle$. Hence, by (v), $b \in \langle z_y \rangle$, so b commutes with z, so $bz^s b^{-1} = z^{2s}$ is impossible.) Thus, $\epsilon_1 = -1$. Therefore, $a_1 t^{\epsilon_2} \dots t^{\epsilon_m} a_m x a_m^{-1} t^{-\epsilon_m} \dots t^{-\epsilon_2} a_1^{-1} = z^s$ for some non-zero $s \in \mathbb{Z}$. Analogously, we get that $\epsilon_2 = 1$, as otherwise z^s would be conjugated in G_α with some power of x, and this would imply that z^s and z^{2s} are conjugated in G_α. We conclude that a_1 conjugates two powers of z, so by (v) for G_α, a_1 must also be a power of z. It is a contradiction, since $t^{\epsilon_1} a_1 t^{\epsilon_2} = t^{-1} a_1 t$ can be reduced, which is impossible by the irreducibility of the word $a_0 t^{\epsilon_1} a_1 \dots t^{\epsilon_m} a_m$.

(iv) Regardless of the full definition of f_β, the partial definition of f_β given above guarantees that (iv) holds for $\alpha_1, \alpha_2 \leq \beta$.

Before we turn to the proof of (v) for G_β, let us prove the following lemma.

Lemma 3.3 *Let $m \in \mathbb{N} \setminus \{0\}$. If $a \in G_{\alpha+1}$, $b \in G_\alpha \setminus \{e\}$ and $ab^m a^{-1} \in G_\alpha$, then $a \in G_\alpha$ or $b \in S_{G_{\alpha+1}}$.*

Proof. Suppose $a \notin G_\alpha$, and write a as a reduced word. Since all letters $t^{\pm 1}$ in the word $ab^m a^{-1}$ must be reduced, b^m is conjugated with a power of x or of z in G_α.

First, consider the case when b^m is conjugated in G_α with x^l for some $l \in \mathbb{Z}$. Then, since x is conjugated with x^2 in G_α, b^m is conjugated with b^{2m} via some element $c \in G_\alpha$. Suppose for a contradiction that $b \notin S_{G_{\alpha+1}}$. Then, by (v) for G_α, we have that $c \in \langle z_b \rangle$, so c commutes with $b^m = z_b^{mn}$, which contradicts the fact that $cb^m c^{-1} = b^{2m}$.

Now, consider the case when b^m is conjugated in G_α with a power of z, i.e., there exists $c \in G_\alpha$ such that $cb^m c^{-1} = z^l$ for some $l \in \mathbb{Z} \setminus \{0\}$. Then $cbc^{-1} \in C(z^l)$. On the other hand, using (v) for G_α, one gets that $z_z = z^{\pm 1}$, where $z_z = f_\alpha(z)$. (Indeed, $z = z_y$ for some $y \in G_\alpha \setminus (S_{G_\alpha} \cup \{e\})$. By (v), $y = z^{n_0}$ for some $n_0 \in \mathbb{N} \setminus \{0\}$, so $z \in G_\alpha \setminus (S_{G_\alpha} \cup \{e\})$. By (v), we have $z = z_z^{n_1}$ for some $n_1 \in \mathbb{N}$, so z_z commutes with $y = z_z^{n_0 n_1}$. Thus, once again by (v), we get that $z_z \in \langle z \rangle$. So, $z = z^{n_1 n}$ for some $n \in \mathbb{Z}$, which implies that $n_1 = \pm 1$.) Thus, by (v) for G_α, we conclude that $cbc^{-1} \in \langle z \rangle$, in particular $b \in S_{G_{\alpha+1}}$. □

(v) Let $y \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$. We will check that since z_y satisfies Condition (v) for y and α, it also satisfies it for y and $\alpha + 1$. It is enough to show that for every $k \in G_{\alpha+1} \backslash G_\alpha$ and $m \in \mathbb{N} \setminus \{0\}$ we have $ky^m k^{-1} \notin \langle z_y \rangle$. If it is not true, then Lemma 3.3 implies that $y \in S_{G_{\alpha+1}} \cup \{e\}$, a contradiction.

Let $y \in G_\alpha^{G_{\alpha+1}} \setminus (S_{G_{\alpha+1}} \cup G_\alpha)$. Then $y = y_\alpha^s$ for some $y_\alpha \in G_\alpha \setminus (S_{G_{\alpha+1}} \cup \{e\})$ and $s \in G_{\alpha+1}$. Define $f_\beta(y) = z_y := z_{y_\alpha}^s$. It follows from the above paragraph and the fact that conjugation is a group automorphism that (v) holds for y and $\alpha + 1$.

The bunch of lemmas below will finally allow us to deal with the remaining case when $y \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$.

Lemma 3.4 *If $a \in G_{\alpha+1} \setminus G_\alpha^{G_{\alpha+1}}$ and $n \in \mathbb{N} \setminus \{0\}$, then $a^n \notin G_\alpha^{G_{\alpha+1}}$.*

Proof. Suppose $a^n \in G_\alpha^{G_{\alpha+1}}$. Conjugating if necessary, we can assume that $a^n \in G_\alpha \setminus \{e\}$. Since $aa^n a^{-1} = a^n \in G_\alpha$, Lemma 3.3 shows that $a^n \in S_{G_{\alpha+1}}$. Then, since $a \in C_{G_{\alpha+1}}(a^n)$, we see that a is conjugated with an element centralizing x which must lie in G_α by (iii), so $a \in G_\alpha^{G_{\alpha+1}}$, a contradiction. □

Lemma 3.5 *If $w, c \in G_{\alpha+1}$, $wc = cw$ and there exists $a \notin G_\alpha^{G_{\alpha+1}}$ commuting with c and w, then there exists $r \in G_{\alpha+1}$ and $k, l \in \mathbb{Z}$ such that $w = r^k, c = r^l$.*

Proof. By $|w|_t$ and $|c|_t$ we denote the total number of letters $t^{\pm 1}$ in reduced words representing w and c, respectively. The proof will be by induction on $|w|_t + |c|_t$.

Base step: Assume that $|w|_t = 0$ or $|c|_t = 0$, wlog $|c|_t = 0$. If $c \neq e$, then Lemma 3.3 implies that $c \in S_{G_{\alpha+1}}$ (otherwise a would be in G_α), so, by (iii), $a \in G_\alpha^{G_{\alpha+1}}$, a contradiction. We have proved that $c = e$, and we can put $r = w$.

Induction step: Wlog we can assume that $|w|_t, |c|_t > 0$, w is cyclically reduced, and w, c are written in normal form. Since w is cyclically reduced, after concatenation either of words c, w or of w, c^{-1} the only possible reduction is a multiplication of two elements from G_α. If there are no reductions of letters $t^{\pm 1}$ between c and w, then, since $cw = wc$, the only possible reduction after concatenation of w and c can be a multiplication of two elements of G_α. If there are no reductions of letters $t^{\pm 1}$ between w and c^{-1}, then, since $c^{-1}w = wc^{-1}$, the only possible reduction after concatenation of c^{-1} and w can be a multiplication of two elements of G_α. Replacing c by c^{-1} if necessary (and writing it in normal form), we have that $wc = cw$ and the only possible reductions after concatenation of w, c and of c, w can be multiplications of two elements of G_α.

Consider the case $|w|_t \geq |c|_t$ (the case $|c|_t > |w|_t$ is handled analogously). By the uniqueness of normal form, the equation $wc = cw$ provides us a division $w_1 w_2$ of the word w and an element $h \in G_\alpha$ such that $w_2 = hc$. Then wc^{-1} commutes with c and a, but $wc^{-1} = w_1 h$, so $|wc^{-1}|_t < |w|_t$. By induction hypothesis, we get $wc^{-1} = r^k, c = r^l$, which implies $w = r^{k+l}$. $\square$

Lemma 3.6 *Let $\zeta \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$, $a \in G_{\alpha+1}$, $n \in \mathbb{N}$ and $a = \zeta^n$. If a is cyclically reduced, then so is ζ.*

Proof. We can write ζ as yby^{-1}, where b is cyclically reduced and such that we can choose a word representing y and a cyclically reduced word $a_0 t^{\epsilon_1} a_1 \ldots t^{\epsilon_m}$ representing b so that the only possible reduction in the word yby^{-1} can be a multiplication of two elements of G_α between y, b. Notice that $b \notin G_\alpha$, as otherwise $\zeta \in G_\alpha^{G_{\alpha+1}}$. So, since b is written as a cyclically reduced word, the only possible reduction in $a = yb^n y^{-1}$ can be a multiplication of two elements from G_α between y, b. Hence, if $|y|_t > 0$, then a is not cyclically reduced, a contradiction. So $y \in G_\alpha$. Consider the case $\epsilon_1 = 1$ and $\epsilon_m = 1$ (the other cases are analogous). Since b is cyclically reduced, the condition that ζ is not cyclically reduced as an element of the group (i.e., all its representations are non cyclically reduced words) is equivalent to the condition $ya_0 \notin \langle z \rangle$ and $y^{-1} \notin \langle x \rangle$. But this condition implies that $a = yb^n y^{-1}$ is not cyclically reduced (as an element of the group), a contradiction. Thus, ζ is cyclically reduced. $\square$

Lemma 3.7 *If $a \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$, then there exist $n \in \mathbb{N}$ and $w \in G_{\alpha+1}$ such that $w^n = a$ and w does not have proper roots in $G_{\alpha+1}$. Such a w we will call a minimal root.*

Proof. In order to prove it, it is enough to find a bound on the degree of roots which we can take of a. Wlog a is cyclically reduced, so, by Lemma 3.6, its every root w is also cyclically reduced. Since $a \notin G_\alpha$, $|w|_t \geq 1$. Hence, the degree of the root w is bounded by $|a|_t$. $\square$

Lemma 3.8 *If $a, b \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$, $n \in \mathbb{N} \backslash \{0\}$ and $a^n = b^n$, then $a = b$.*

Proof. Wlog we can assume that a is cyclically reduced and its cyclically reduced (and normal) representation ends with the letter t. Then a^n is also cyclically reduced, so, by Lemma 3.6, b is cyclically reduced. Hence, $|a|_t = |b|_t$. By the uniqueness of normal form and the fact that the normal form of a ends with t, the equation $a^n = b^n$ implies that there exists $p \in \mathbb{Z}$ such that $a = x^p b$. Analogously, by the uniqueness of normal form for the term $a^{-n} = b^{-n}$, one can show that there exists $m \in \mathbb{Z}$ such that $a = bx^m$. Now, if $p = 0$ or $m = 0$, we are done. So suppose $p, m \neq 0$. Then x^p and x^m are conjugated by some $c \in G_0$, i.e., $cx^m c^{-1} = x^p$. Thus, $bc^{-1} \in C_{G_{\alpha+1}}(x^p)$ and $p \neq 0$, so, by (iii), we have $bc^{-1} \in G_0$. Hence, $b \in G_0$, a contradiction. $\square$

Lemma 3.9 *Assume that $a, b \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$ and $n, m \in \mathbb{N}\backslash\{0\}$, and a, b do not have proper roots in $G_{\alpha+1}$. If $a^n = b^m$, then $a = b$ and $m = n$.*

Proof. Wlog a is cyclically reduced and and its cyclically reduced (and normal) representation ends with the letter t. Then, by Lemma 3.6, b is also cyclically reduced. By the uniqueness of normal form applied to $a^n = b^m$, one gets that the normal form of b also ends with t. The proof will be by induction on the number of letters $t^{\pm 1}$ in a shorter of the words a, b. Let $k = |a|_t$ and $l = |b|_t$. Since a, b are cyclically reduced, $kn = lm$.

Base step: Assume $k = 1$ (the case $l = 1$ is handled in the same way). Then $n = lm$. So $(a^l)^m = b^m$. By assumption and Lemma 3.4, $a^l, b \in G_{\alpha+1} \backslash G_\alpha^{G_{\alpha+1}}$. Thus, Lemma 3.8 implies that $a^l = b$. Since b has no proper roots, we conclude that $l = 1$, so $n = m$ and $a = b$.

Induction step: Assume $k \leq l$ (the case $k > l$ is handled similarly). We can assume that $d := \gcd(k, l) < k$, because otherwise the equation $kn = ml$ implies $m|n$, and we get the desired conclusion as in the base step. By Euclid's algorithm, there exist $p, q \in \mathbb{Z}$ such that $pk + ql = d$, $p > 0$ and $q < 0$. Since $a, b \in C_{G_{\alpha+1}}(a^n)$, also $a^p b^q \in C_{G_{\alpha+1}}(a^n)$. By Lemma 3.4, $a^n \notin G_\alpha^{G_{\alpha+1}}$. Therefore, by Lemma 3.5, we get that $a^p b^q = r^{h_1}, a^n = r^{h_2}$ for some $r \in G_{\alpha+1}$ and $h_1, h_2 \in \mathbb{Z}$. Moreover, we can assume that r has no proper root by Lemma 3.7. By Lemma 3.6, r is cyclically reduced. On the other hand, by the uniqueness of normal form applied to $a^n = b^m$ and the fact that the normal form of b ends with t, we get that b^{-q} is a suffix of a^p modulo x^i on the left side, and so $|a^p b^q|_t = pk + ql = d$. By the above observations, $|r|_t \leq d < k$, so the equation $a^n = r^{h_2}$ together with the induction hypothesis give us that $a = r^{\pm 1}$, and the equation $b^m = r^{h_2}$ that $b = r^{\pm 1}$. Since r has infinite order, $a = b$ and $m = n$. □

Now, we are ready to complete the proof of Condition (v), i.e., we consider the case when $y \in G_{\alpha+1} \setminus G_\alpha^{G_{\alpha+1}}$. By Lemmas 3.7 and 3.9, there exists a unique minimal root z_y of y; so $z_y^k = y$ for some $k \in \mathbb{N}$. We define $f_\beta(y) := z_y$. Wlog we can assume that y is cyclically reduced. By Lemma 3.6, z_y is also cyclically reduced. Let c be such that $cy^m c^{-1} = z_y^n$ for some $m \in \mathbb{N} \setminus \{0\}$ and $n \in \mathbb{Z}$. Raising this equation to a big power and counting occurrences of the letters $t^{\pm 1}$ on both sides, we easily get that $n = \pm mk$. Hence, for $d = c$ or $d = c^2$ we have $dy^m = y^m d$. By Lemma 3.4 and the assumption that $y \notin G_\alpha^{G_{\alpha+1}}$, we get $y^m \notin G_\alpha^{G_{\alpha+1}}$. Thus, Lemma 3.5 gives us that $d = r^{h_1}$ and $y^m = r^{h_2}$ for some $r \in G_{\alpha+1}$ and $h_1, h_2 \in \mathbb{Z}$. By Lemma 3.7, we can assume that r does not have proper roots. Since $y^m = r^{h_2}$ and $y^m \notin G_\alpha^{G_{\alpha+1}}$, we have $r \notin G_\alpha^{G_{\alpha+1}}$. The equation $z_y^{mk} = r^{h_2}$ and Lemma 3.9 imply that $r = z_y^{\pm 1}$. In the case when $d = c$, this shows that $c \in \langle z_y \rangle$. Consider the case $d = c^2$. Since $d = r^{h_1}$ and $r \notin G_\alpha^{G_{\alpha+1}}$, by Lemma 3.4, $d \notin G_\alpha^{G_{\alpha+1}}$, and so $c \notin G_\alpha^{G_{\alpha+1}}$. Thus, c has a minimal root c_0, say $c_0^l = c$. The equality $c_0^{2l} = r^{h_1}$ together with Lemma 3.9 yield $c_0 = r^{\pm 1}$, so $c \in \langle z_y \rangle$.

Now, we will describe how to choose the elements z at the "even" steps of the construction in order to satisfy Condition (ii). We always take this element $z \in Z_\alpha$ which was created as early as possible during the construction. Since all G_α's are countable, using the definition of Z_α's and the fact $z_z = z^{\pm 1}$ for $z \in Z_\alpha$ (see the proof of Lemma 3.3), one easily gets that for each $\alpha < \omega_1$ after countably many steps all elements from Z_α will be conjugated with x. Now, if $e \neq y \in G_\alpha \backslash S_{G_\alpha}$, then by (v), $z_y^n = y$ for some $n \in \mathbb{N} \setminus \{0\}$. But, as $z_y \in Z_\alpha$, we already know that there exists $t \in G$ such that $txt^{-1} = z_y$. Hence, $tx^n t^{-1} = y$. From the choice of G_0, we know that $x^n \in S_{G_0}$, so $y \in S_G$. Thus, all non-trivial elements of G are conjugated with x.

As it was mentioned in the previous section, we finish the paper with an example of a (non-pure) minimal group G whose theory T has quantifier elimination, there is a model $G_1 \succ G$ which is quasi-minimal, and the global generic type of G (so also of G_1) is not generically stable; thus, both G and G_1 are not generically stable. Moreover, $\mathrm{Th}(G)$ has NIP.

For the rest of the paper, $G = (F, 0, +, <, P_n)_{n \in \omega}$, where:

- $(F, +, 0)$ is the group of exponent 2 with neutral element 0, spanned freely over $\mathbb{Z}_2$ by $(e_i : i \in \omega)$, i.e., $F = \bigoplus_{i \in \omega} \mathbb{Z}_2 e_i$. In other words, F is the subgroup of $\mathbb{Z}_2^\omega$ consisting of the elements with almost all coordinates equal to zero.

- For $a \neq 0$, we define

$$a < b \iff \max\{i \in \omega : \pi_i(a) = 1\} < \max\{i \in \omega : \pi_i(b) = 1\},$$

  where π_i is the projection on the i-th coordinate. Moreover, $0 < b \iff b \neq 0$.

- For $n \neq 0$, we define

$$P_n(a, b) \iff (a < b \wedge n = \max\{k \in \omega : \exists x_1, \ldots, x_k (a < x_1 < \cdots < x_k < b)\}).$$

  Moreover, $P_0(a, b)$ means that $a < b$ and there is no c with $a < c < b$.

It is clear that all P_n's are definable using $<$.

The proofs of the next lemmas are left as exercises.

Lemma 3.10 *The following list of axioms (which are all first order) axiomatizes* $\mathrm{Th}(G)$.

1. *$+$ is a group law with neutral element 0, giving a group of exponent 2.*

2. $(\forall x \neq 0)(0 < x)$.

3. *The definitions of P_n's.*

4. *The formula $\neg x < y \wedge \neg y < x$ defines an equivalence relation, which will be denoted by $\sim$.*

5. $(\forall x, y, z)((x \sim y \wedge x < z) \longrightarrow y < z)$ *and* $(\forall x, y, z)((x \sim y \wedge z < x) \longrightarrow z < y)$. *So, we can define* $<^*$ *on the quotient sort by* $[x]_\sim <^* [y]_\sim \iff x < y$.

6. $<^*$ *is a linear order with the smallest element* $[0]_\sim$ *and such that* $[0]_\sim$ *has an immediate successor, and any element different from* $[0]_\sim$ *has an immediate successor and an immediate predecessor.*

7. $(\forall x, y)(x < y \longrightarrow x + y \sim y)$ *and* $(\forall x, y)(x \sim y \longrightarrow x + y < x)$.

Lemma 3.11 Th(G) *has quantifier elimination.*

The following corollary follows easily by quantifier elimination (to check NIP, one should use the fact that for theories with quantifier elimination it is enough to prove NIP for atomic formulas $\varphi(x, \overline{y})$, where x is a single variable [1, Proposition 12]).

Corollary 3.12 *G is minimal and has NIP.*

The underlying order witnesses that a Morley sequence in the global generic type is not totally indiscernible, so the generic type is not generically stable.

Now, our goal is to construct a quasi-minimal model of Th(G). Notice that the order $(\omega_1, <)$ is not quasi-minimal, because the limit ordinals form a definable set which is neither countable nor co-countable. We will use a certain quasi-minimal modification of this order as a "base" of our structure. Namely, let I be the order obtained from $(\omega_1, <)$ be replacing each infinite ordinal by a copy of $(\mathbb{Z}, <)$. Let G_1 be the group of exponent 2 spanned freely over $\mathbb{Z}_2$ by I. For $a, b \in G_1 \setminus \{0\}$, we define $a < b \iff \max\{i \in I : \pi_i(a) = 1\} < \max\{i \in I : \pi_i(b) = 1\}$, where π_i is the projection on the i-th coordinate. Moreover, put $0 < b \iff b \neq 0$. P_n's are defined as in G. Then G is a substructure of G_1. It is clear that G_1 satisfies Axioms (1)-(7) from Lemma 3.10, so $G_1 \models$ Th(G), and, by quantifier elimination, $G_1 \succ G$. By the choice of I, a similar argument to the proof of Corollary 3.12 yields the following

Corollary 3.13 *G_1 is quasi-minimal.*

Question 3.14 *Does every minimal structure in a countable language posses a quasi-minimal elementary extension?*

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