

RAMSEY THEORY AND TOPOLOGICAL DYNAMICS FOR FIRST ORDER THEORIES

KRZYSZTOF KRUPIŃSKI, JUNGUK LEE, AND SLAVKO MOCONJA

ABSTRACT. We investigate interactions between Ramsey theory, topological dynamics, and model theory. We introduce various Ramsey-like properties for first order theories and characterize them in terms of the appropriate dynamical properties of the theories in question (such as [extreme] amenability of a theory or some properties of the associated Ellis semigroups). Then we relate them to profiniteness and triviality of the Ellis groups of first order theories. In particular, we find various criteria for [pro]finiteness and for triviality of the Ellis group of a given theory from which we obtain wide classes of examples of theories with [pro]finite or trivial Ellis groups. We also find several concrete examples illustrating the lack of implications between some fundamental properties. In the appendix, we give a full computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary relation. This example shows that the assumption of NIP in the version of Newelski's conjecture for amenable theories (proved in [15]) cannot be dropped.

1. INTRODUCTION

In their seminal paper [14], Kechris, Pestov and Todorčević discovered surprising interactions between dynamical properties of the group of automorphisms of a Fraïssé structure and Ramsey-theoretic properties of its age. For example, they proved that this group is extremely amenable iff the age has the structural Ramsey property and consists of rigid structures (equivalently, the age has the embedding Ramsey property in the terminology used by Zucker in [31]). This started a wide area of research of similar phenomena. Recently, Pillay and the first author [17] gave a model-theoretic account for the fundamental results of Kechris-Pestov-Todorčević (shortly KPT) theory, generalizing the context to arbitrary, possibly uncountable, structures. However, KPT theory (including such generalizations) is not really about model-theoretic properties of the underlying theory, because: on the dynamical side, it talks about the topological dynamics of the topological group of automorphisms of a given structure, which can be expressed in terms of the action of this group on the universal ambit rather than on type spaces of the underlying theory, and, on the Ramsey-theoretic side, it considers arbitrary colorings (without any definability properties) of the finite subtuples of a given model. Definitions of Ramsey properties for a given structure stated in [17] suggest the corresponding

2010 *Mathematics Subject Classification.* 03C45, 05D10, 54H20, 54H11, 20E18.

Key words and phrases. Ramsey property, Ramsey degree, Ellis group, [extremely] amenable theory, profinite group.

All authors are supported by National Science Center, Poland, grant 2016/22/E/ST1/00450. The first author is also supported by National Science Center, Poland, grant 2018/31/B/ST1/00357. The third author is also supported by the Ministry of Education, Science and Technological Development of Serbia.

definitions for first order theories just by applying them to a monster model. In this paper, we go much further and define various “definable” versions of Ramsey properties for first order theories by restricting the class of colorings to “definable” ones. And then we find the appropriate dynamical characterizations of our “definable” Ramsey properties in terms of the dynamics of the underlying theory (in place of the dynamics of the group of automorphisms of a given model) some of which are surprising and different comparing to classical KPT theory.

The classes of amenable and extremely amenable theories introduced and studied in [12] are defined in a different way than typical Shelah-style, combinatorially defined classes of theories (such as NIP, simple, NTP_2). In this paper, we give Ramsey-theoretic characterizations of [extremely] amenable theories; these characterizations are clearly combinatorial, but still of different flavor than Shelah’s definitions. Also, the new classes of theories introduced in this paper via some Ramsey-theoretic properties or via their dynamical characterizations do not follow the usual Shelah-style way of defining new classes of theories. This makes the whole topic rather novel in model theory.

We find the interaction between “definable” Ramsey properties and the dynamics of first order theories natural and interesting in its own right. However, our original motivation to introduce the “definable” Ramsey properties has some specific origins in model theory and topological dynamics in model theory, which we explain in the next paragraph.

Some methods of topological dynamics were introduced to model theory by Newelski in [22]. Since then a wide research on this topic has been done by Chernikov, Hrushovski, Newelski, Pillay, Rzepecki, Simon, the first author, and others. For any given theory T , a particularly important place in this research is reserved for the investigation of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where \bar{c} is an enumeration of a monster model $\mathfrak{C} \models T$ and $S_{\bar{c}}(\mathfrak{C})$ is the space of global types extending $\text{tp}(\bar{c}/\emptyset)$, as it turns out that topological properties of this flow carry important information about the underlying theory. In particular, in [18] it was proved that there exists a topological quotient epimorphism from the Ellis group of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ (also called the Ellis group of T , as it does not depend on the choice of the monster model by [15]) to $\text{Gal}_{\text{KP}}(T)$ (the Kim-Pillay Galois group of T), and even to the larger group $\text{Gal}_{\text{L}}(T)$ (the Lascar Galois group of T); in particular, the Ellis group of T captures more information about T than the Galois groups of T . This was the starting point for the research in this paper. Namely, from the aforementioned result from [18] one easily deduces that profiniteness of the Ellis group implies profiniteness of $\text{Gal}_{\text{KP}}(T)$, which in turn is known to be equivalent to the equality of the Shelah and Kim-Pillay strong types. The question for which theories the Shelah and Kim-Pillay strong types coincide is fundamental in model theory (note that it can be viewed as “elimination of bounded hyperimaginaries”; in the context of simple theories, it is equivalent to the independence theorem being true over the (imaginary rather than hyperimaginary) algebraically closed sets). This is known to be true in e.g. stable or supersimple theories, but remains a well-known open question in simple theories in general. This led us to the question for which theories the Ellis group is profinite, which is also interesting in its own rights (keeping in mind that the Ellis group of T captures more information than any of the Galois groups of T). And among the main outcomes of this paper are results saying that various Ramsey-like properties of T imply profiniteness of the Ellis group.

Let us briefly discuss the Ramsey properties which we investigate in this paper. They are given with respect to a monster model \mathfrak{C} of a first-order theory T , but we will show that they do not depend on the choice of \mathfrak{C} , so they are really properties of T . We say that T has *separately finite elementary embedding Ramsey degree* (*sep. fin. EERdeg*) if for every finite $\bar{a} \subseteq \mathfrak{C}$ there exists $l < \omega$ such that for every finite $\bar{b} \supseteq \bar{a}$, $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$. Here, for a tuple \bar{a} and a set B (or a tuple which is treated as the set of coordinates), $\binom{B}{\bar{a}}$ denotes the set of all $\bar{a}' \subseteq B$ such that $\bar{a}' \equiv \bar{a}$. If l above can be taken to be 1 for every finite \bar{a} , we say that T has the *elementary embedding Ramsey property* (*ERP*). If l can be taken to be 1 and we restrict ourselves to considering only [externally] definable colorings (see Section 3 for definitions), we say that T has the [*externally*] *definable elementary embedding Ramsey property* (*[E]DEERP*). If for every finite set of formulae Δ and every finite \bar{a} the above holds (for some l) for the externally definable Δ -colorings, then we say that T has *separately finite externally definable elementary embedding Ramsey degree* (*sep. fin. EDEERdeg*).

Theories with *ERP* and *sep. fin. EERdeg* are generalizations of the classical notions of embedding Ramsey property and finite embedding Ramsey degree in the following sense: If K is an \aleph_0 -saturated Fraïssé structure, then its age has the embedding Ramsey property [*sep. fin. embedding Ramsey degree*] iff $\text{Th}(K)$ has *ERP* [*sep. fin. EERdeg*].

We also consider the following convex Ramsey-like properties. We say that T has the *elementary embedding convex Ramsey property* (*EECRP*) if for every $\epsilon \geq 0$ and finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \text{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ the convex combinations $\sum_{j < k} \lambda_j c(\sigma_j(\bar{a}'))(i)$ and $\sum_{j < k} \lambda_j c(\sigma_j(\bar{a}''))(i)$ differ by at most ϵ for every $i < n$. If we restrict ourselves to definable colorings, we say that T has the *definable elementary embedding convex Ramsey property* (*DEECRP*).

To state our main results, we need to use a natural refinement of the usual space of Δ -types, denoted by $S_{\bar{c}, \Delta}(\bar{p})$ for a finite set of formulae $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ and a finite set (or sequence) of types $\bar{p} = \{p_j(\bar{y})\}_{j < m} \subseteq S_{\bar{y}}(\emptyset)$. (It is defined after Lemma 2.20.) For a flow (G, X) , by $EL(X)$ we denote the Ellis semigroup of this flow. By $\text{Inv}_{\bar{c}}(\mathfrak{C})$, we denote the space of global invariant types extending $\text{tp}(\bar{c}/\emptyset)$. (All these notations and definitions can be found in Section 2.) Our main result yields dynamical characterizations of the introduced Ramsey properties.

Theorem 1. *Let T be a complete first-order theory and \mathfrak{C} its monster model. Then:*

- (i) *T has DEERP iff T is extremely amenable (in the sense of [12]).*
- (ii) *T has EDEERP iff there exists $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$.*
- (iii) *T has sep. fin. EDEERdeg iff for every finite set of formulae Δ and finite sequence of types \bar{p} there exists $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ such that $\text{Im}(\eta)$ is finite.*
- (iv) *T has DEECRP iff T is amenable (in the sense of [12]).*

A relation of this to the Ellis group of the theory is given by the next corollary.

Corollary 2. (i) *A theory with EDEERP has trivial Ellis group (see Cor. 3.15).*
(ii) *A theory with sep. fin. EDEERdeg has profinite Ellis group (see Cor. 4.1).*

Item (i) is an easy consequence of Theorem 1(ii). Item (ii) follows from Theorem 1(iii) and the implication (D) \implies (A) in Theorem 3 below.

In the next theorem, \mathcal{M} denotes a minimal left ideal in $\text{EL}(S_{\bar{c}}(\mathfrak{C}))$ and u an idempotent in this ideal, so $u\mathcal{M}$ is the Ellis group of T ; $u\mathcal{M}/H(u\mathcal{M})$ is the canonical Hausdorff quotient of $u\mathcal{M}$ (see Section 2). Analogously, $u_{\Delta, \bar{p}}\mathcal{M}_{\Delta, \bar{p}}$ is the Ellis group of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}, \Delta}(\bar{p}))$. The main idea behind the next result is that a natural way to obtain that the Ellis group of T is profinite is to present the flow $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of some flows each of which has finite Ellis group, and if it works, it should also work for the natural presentation of $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of the flows $S_{\bar{c}, \Delta}(\bar{p})$ (where Δ and \bar{p} vary).

Theorem 3. *Consider the following conditions:*

- (A'') $\text{Gal}_{\text{KP}}(T)$ is profinite;
- (A') $u\mathcal{M}/H(u\mathcal{M})$ is profinite;
- (A) $u\mathcal{M}$ is profinite;
- (B) the $\text{Aut}(\mathfrak{C})$ -flow $S_{\bar{c}}(\mathfrak{C})$ is isomorphic to the inverse limit $\varprojlim_{i \in I} X_i$ of some $\text{Aut}(\mathfrak{C})$ -flows X_i each of which has finite Ellis group;
- (C) for every finite sets of formulae Δ and types $\bar{p} \subseteq S(\emptyset)$, $u_{\Delta, \bar{p}}\mathcal{M}_{\Delta, \bar{p}}$ is finite;
- (D) for every finite sets of formulae Δ and types $\bar{p} \subseteq S(\emptyset)$, there exists $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ with $\text{Im}(\eta)$ finite.

Then (D) \implies (C) \iff (B) \implies (A) \implies (A') \implies (A'').

We also find several other criteria for [pro]finiteness of the Ellis group. Applying Corollary 2 or our other criteria together with some well-known theorems from structural Ramsey theory (saying that various Fraïssé classes have the appropriate Ramsey properties), we get wide classes of examples of theories with [pro]finite or sometimes even trivial Ellis groups. But we also find some specific examples illustrating interesting phenomena, e.g. we give examples showing that in Theorem 3: (A'') does not imply (A'), and (A') does not imply (B). The example showing that (A'') does not imply (A') is supersimple of SU-rank 1, so it shows that even for supersimple theories the Ellis group of the theory need not be profinite. We have not found examples showing that (C) does not imply (D), and (A) does not imply (B), which we leave as open problems.

One of our most important examples is Example 5.7 which is analyzed in details in the appendix: we give there a precise computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary relation. This group turns out to be $\mathbb{Z}/2\mathbb{Z}$. This example is interesting for various reasons. Firstly, by classical KPT theory, we know that it has sep. finite $EERdeg$, so the Ellis group is profinite by the above results (in fact, it satisfies the assumptions of some other criteria that we found, which implies that the Ellis group is finite), and the example shows that it may be non-trivial. A variation of this example (see Example 5.8) yields an infinite Ellis group, which shows that in some of our criteria for profiniteness, we cannot expect to get finiteness of the Ellis group. Example 5.7 also shows that in general sep. fin. $EERdeg$ does not imply $EDEERP$ and that $DEERP$ does not imply $EDEERP$. Finally, this example is easily seen to be extremely amenable in the sense of [12], so its KP-Galois group is trivial. But the Ellis group is non-trivial. Hence, the epimorphism (found in [18]) from the Ellis group to the KP-Galois group is not an isomorphism. On the other hand, by [15, Theorem 0.7], we know that under NIP, even amenability of the theory is sufficient for this epimorphism to be an isomorphism. So our example shows that one cannot drop the NIP assumption in [15, Theorem 0.7], which was not known so far.

Using our observations that both properties *EERP* and *EECRP* do not depend on the choice of the monster model, or even an \aleph_0 -saturated and strongly \aleph_0 -homogeneous model $M \models T$, and the results from [17] saying that *EERP* (defined in terms of M) is equivalent to extreme amenability of the topological group $\text{Aut}(M)$, and *EECRP* (defined in terms of M) is equivalent to amenability of $\text{Aut}(M)$, we get the following corollary.

Corollary 4. *Let T be a complete first-order theory. The group $\text{Aut}(M)$ is [extremely] amenable as a topological group for some \aleph_0 -saturated and strongly \aleph_0 -homogeneous model $M \models T$ iff it is [extremely] amenable as a topological group for all \aleph_0 -saturated and strongly \aleph_0 -homogeneous models $M \models T$.*

This means that [extreme] amenability of the group of automorphisms of an \aleph_0 -saturated and strongly \aleph_0 -homogeneous structure is actually a property of its theory, which seems to be a new observation.

The paper is organized as follows. In Section 2, we recall or introduce all the necessary notions from model theory, topological dynamics, and classical structural Ramsey theory. Furthermore, we prove several new fundamental and useful observations, which for example are helpful to compute Ellis groups or show some dynamical properties of concrete theories. Section 3 is the central part of the paper. We introduce and characterize all the aforementioned Ramsey properties for first order theories. We prove Theorem 1 and Corollary 4. In Section 4, we prove Theorem 3 and find some other conditions which imply [pro]finiteness of the Ellis group of the theory. In Section 5, we give a long list of examples to which our results apply, and find several examples with some specific properties, e.g. the aforementioned examples showing the lack of two of the implications between the items of Theorem 3. In the appendix, we give a complete computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary symbol.

Some “definable” versions of Ramsey properties were also introduced and considered in a recent paper by Nguyen Van Thé [24]. Also, Hrushovski [11] has recently introduced some version of Ramsey properties in a first-order setting. But all these notions seem to be different and they are introduced for different reasons. It would be interesting to see in the future if there are any relationships.

2. PRELIMINARIES AND FUNDAMENTAL OBSERVATIONS

Most of this section consists of definitions, notations and facts needed in this paper. But there are also new ingredients, especially in Subsection 2.4, where we obtain some new reductions and introduce the type spaces $S_{\Delta, \bar{c}}(\bar{p})$ playing a key role in this paper.

2.1. Model theory. We use standard model-theoretic concepts and terminology. By a theory we always mean a complete first-order theory T in a language L . For simplicity, we will be assuming that L is one-sorted, but the whole theory developed in this paper works almost the same for many-sorted languages. We usually work in a monster model \mathfrak{C} of T , i.e. a κ -saturated and strongly κ -homogeneous model of T for a large enough cardinal κ (called the degree of saturation of \mathfrak{C}). Elements of \mathfrak{C} are denoted by a, b, \dots and tuples (finite or infinite) of \mathfrak{C} are denoted by \bar{a}, \bar{b}, \dots . By a small set [model] we mean a subset [elementary submodel] of \mathfrak{C} of cardinality

less than κ ; small subsets of \mathfrak{C} are denoted by A, B, \dots , and small submodels by M, N, \dots .

Global types are complete types over \mathfrak{C} . By $S_{\bar{x}}(A)$ we denote the space of all complete types over A in variables \bar{x} ; if $A = \emptyset$, we also write $S_{\bar{x}}(T)$ for $S_{\bar{x}}(\emptyset)$. For a type $\pi(\bar{x})$ over some $B \subseteq A$, $S_{\pi}(A)$ denotes the subspace of $S_{\bar{x}}(A)$ consisting of all types extending $\pi(\bar{x})$. For a tuple \bar{a} , $S_{\bar{a}}(A)$ denotes the space of all complete types over A extending $\text{tp}(\bar{a})$; in other words, $S_{\bar{a}}(A) = S_{\text{tp}(\bar{a})}(A)$. These spaces are naturally compact, Hausdorff, 0-dimensional topological spaces. For tuples \bar{a}, \bar{b} , $\bar{a} \equiv \bar{b}$ means that \bar{a} and \bar{b} have the same type over \emptyset .

$\text{Aut}(\mathfrak{C})$ is the group of all automorphisms of \mathfrak{C} , and $\text{Aut}(\mathfrak{C}/A)$ is the pointwise stabilizer of A . A subset of a power of \mathfrak{C} is invariant [A -invariant] if it is invariant under $\text{Aut}(\mathfrak{C})$ [$\text{Aut}(\mathfrak{C}/A)$]. Having the same type over \emptyset [small A] is the equivalence relation of lying in the same orbit of $\text{Aut}(\mathfrak{C})$ [$\text{Aut}(\mathfrak{C}/A)$] on the appropriate power of \mathfrak{C} . By \equiv_{Sh} we denote the intersection of all \emptyset -definable, finite equivalence relations (i.e. with finitely many classes) on a given power of \mathfrak{C} ; the classes of \equiv_{Sh} are called Shelah strong types. By $\text{Autf}_{\text{Sh}}(\mathfrak{C})$ we denote the group of all Shelah strong automorphisms of \mathfrak{C} , i.e. all automorphisms of \mathfrak{C} fixing all Shelah strong types. An equivalence relation is bounded if it has less than κ classes. \equiv_{KP} and \equiv_{L} are respectively the finest bounded \emptyset -type-definable equivalence relation and the finest bounded \emptyset -invariant equivalence relation (on a fixed power of \mathfrak{C}); the classes of \equiv_{KP} and \equiv_{L} are called Kim-Pillay strong types and Lascar strong types, respectively. By $\text{Autf}_{\text{KP}}(\mathfrak{C})$ and $\text{Autf}_{\text{L}}(\mathfrak{C})$ we denote respectively the groups of all Kim-Pillay strong automorphism and all Lascar strong automorphisms, i.e. automorphisms of \mathfrak{C} fixing all \equiv_{KP} -classes and all \equiv_{L} -classes, respectively. It turns out that \equiv_{Sh} , \equiv_{KP} , and \equiv_{L} are the orbit equivalence relations of $\text{Autf}_{\text{Sh}}(\mathfrak{C})$, $\text{Autf}_{\text{KP}}(\mathfrak{C})$, and $\text{Autf}_{\text{L}}(\mathfrak{C})$, respectively.

$\text{Autf}_{\text{Sh}}(\mathfrak{C})$, $\text{Autf}_{\text{KP}}(\mathfrak{C})$, $\text{Autf}_{\text{L}}(\mathfrak{C})$ are normal subgroups of $\text{Aut}(\mathfrak{C})$, and the corresponding quotients do not depend on the choice of the monster \mathfrak{C} and are called respectively the Shelah Galois group, the Kim-Pillay Galois group, and the Lascar Galois group of T ; we denote them by $\text{Gal}_{\text{Sh}}(T)$, $\text{Gal}_{\text{KP}}(T)$, and $\text{Gal}_{\text{L}}(T)$, respectively. Since $\text{Autf}_{\text{L}}(\mathfrak{C}) \leq \text{Autf}_{\text{KP}}(\mathfrak{C}) \leq \text{Autf}_{\text{Sh}}(\mathfrak{C})$, we have natural epimorphisms $\text{Gal}_{\text{L}}(T) \rightarrow \text{Gal}_{\text{KP}}(T) \rightarrow \text{Gal}_{\text{Sh}}(T)$.

All the above Galois groups of T are topological groups. The topology on $\text{Gal}_{\text{L}}(T)$ is defined as follows. Let M be a small model and let \bar{m} be an enumeration of M . The natural projection $\text{Aut}(\mathfrak{C}) \rightarrow \text{Gal}_{\text{L}}(T)$ factors through $S_{\bar{m}}(M)$: $\text{Aut}(\mathfrak{C}) \rightarrow S_{\bar{m}}(M) \rightarrow \text{Gal}_{\text{L}}(T)$, and we equip $\text{Gal}_{\text{L}}(T)$ with the quotient topology induced by $S_{\bar{m}}(M) \rightarrow \text{Gal}_{\text{L}}(T)$. This does not depend on the choice of the model M . It turns out that $\text{Gal}_{\text{L}}(T)$ is a compact (but not necessarily Hausdorff) topological group. The topologies on $\text{Gal}_{\text{Sh}}(T)$ and $\text{Gal}_{\text{KP}}(T)$ are defined in similar fashion. $\text{Gal}_{\text{Sh}}(T)$ is a profinite group, and $\text{Gal}_{\text{KP}}(T)$ is a compact, Hausdorff group. The epimorphisms $\text{Gal}_{\text{L}}(T) \rightarrow \text{Gal}_{\text{KP}}(T) \rightarrow \text{Gal}_{\text{Sh}}(T)$ are topological quotient maps.

Furthermore, $\text{Gal}_{\text{Sh}}(T)$ is the largest profinite quotient of $\text{Gal}_{\text{KP}}(T)$. Thus, $\text{Gal}_{\text{KP}}(T)$ is profinite iff it equals $\text{Gal}_{\text{Sh}}(T)$, and the last condition is clearly equivalent to saying that \equiv_{KP} and \equiv_{Sh} are equal on all powers of \mathfrak{C} , i.e. the Kim-Pillay and Shelah strong types coincide.

For more details concerning strong types and Galois groups the reader is referred to [4], [30], or [25, Chapter 2.5].

2.2. Topological dynamics. We quickly introduce and state some facts from topological dynamics. As a general reference we can recommend [1] and [7].

By a G -flow we mean a pair (G, X) where G is a topological group acting continuously on a (non-empty) compact, Hausdorff space X . The *Ellis semigroup* of a G -flow (G, X) , denoted by $\text{EL}(X)$, is the closure of the set $\{\pi_g \mid g \in G\}$ in X^X (equipped with the topology of pointwise convergence), where π_g is the function given by $x \mapsto gx$, with composition as semigroup operation; this semigroup operation is continuous in the left coordinate. $\text{EL}(X)$ itself is a G -flow, where the action is defined by $g\eta = \pi_g \circ \eta$ for $g \in G$ and $\eta \in \text{EL}(X)$. By abusing notation, we denote π_g simply by g , treat G as a subset of $\text{EL}(X)$ (although this ‘inclusion’ is not necessarily 1-1), and then $\text{EL}(X) = \text{cl}(G)$. The minimal G -subflows of $\text{EL}(X)$ coincide with the minimal left ideals of $\text{EL}(X)$. If \mathcal{M} is any minimal left ideal of $\text{EL}(X)$, then the set $\mathcal{J}(\mathcal{M})$ of all idempotents in \mathcal{M} is non-empty. Furthermore, \mathcal{M} is a disjoint union of subsets $u\mathcal{M}$ for $u \in \mathcal{J}(\mathcal{M})$. For each $u \in \mathcal{J}(\mathcal{M})$, $u\mathcal{M}$ is a group with respect to the composition of functions (a subgroup of $\text{EL}(X)$) with neutral u . Moreover, the isomorphism type of this group does not depend on the choice of \mathcal{M} and $u \in \mathcal{J}(\mathcal{M})$, and it is called the *Ellis group* of the flow (G, X) ; abusing terminology, any $u\mathcal{M}$ is also called an (or the) Ellis group of (G, X) . Note that for every $\eta \in \mathcal{M}$, $\eta\mathcal{M} = u\mathcal{M}$ where u is the unique idempotent with $\eta \in u\mathcal{M}$.

The existence of an element in the Ellis semigroup with finite image will be one of the key properties in this paper. The fundamental observation in this situation is given by the next fact, which follows from Lemmas 4.2 and 4.3 in [15]. The idea of the proof of this fact is that if $\eta \in \text{EL}(X)$ has finite image, then for any $u \in \mathcal{J}(\mathcal{M})$ (where \mathcal{M} is a minimal left ideal in $\text{EL}(X)$), the element $u\eta u \in u\mathcal{M}$ also has finite image, and so has u . Then one checks that the map $u\mathcal{M} \rightarrow \text{Sym}(\text{Im}(u))$ given by the restriction $\tau \mapsto \tau|_{\text{Im}(u)}$ is a monomorphism, so $u\mathcal{M}$ is finite.

Fact 2.1. *If there exists $\eta \in \text{EL}(X)$ with $\text{Im}(\eta)$ finite, then the Ellis group is also finite.* \square

On an Ellis group $u\mathcal{M}$ we have a topology inherited from $\text{EL}(X)$. Besides this topology, a coarser, so-called τ -topology is defined. First, for $a \in \text{EL}(X)$ and $B \subseteq \text{EL}(X)$ let $a \circ B$ be the set of all limits of the nets $(g_i b_i)_i$ such that $g_i \in G$, $b_i \in B$ and $\lim_i g_i = a$. For $B \subseteq u\mathcal{M}$ we define $\text{cl}_\tau(B) = u\mathcal{M} \cap (a \circ B)$. cl_τ is a closure operator on $u\mathcal{M}$; the τ -topology is a topology on $u\mathcal{M}$ induced by cl_τ . $u\mathcal{M}$ with the τ -topology is a compact, T_1 semitopological group (i.e. group operation is separately continuous). The isomorphism types of the Ellis groups (for all \mathcal{M} and $u \in \mathcal{J}(\mathcal{M})$) as semitopological groups do not depend on the choice of u and \mathcal{M} . Put $H(u\mathcal{M}) = \bigcap_U \text{cl}_\tau(U)$, where the intersection is taken over all τ -open neighbourhoods of u in $u\mathcal{M}$. This is a τ -closed normal subgroup of $u\mathcal{M}$, and the quotient $u\mathcal{M}/H(u\mathcal{M})$ is a compact, Hausdorff topological group. Moreover, $H(u\mathcal{M})$ is the smallest τ -closed normal subgroup of $u\mathcal{M}$ such that $u\mathcal{M}/H(u\mathcal{M})$ is Hausdorff. $u\mathcal{M}/H(u\mathcal{M})$ will be called the *canonical Hausdorff quotient* of $u\mathcal{M}$.

Remark 2.2. (a) If $u\mathcal{M}/H(u\mathcal{M})$ is profinite, then $u\mathcal{M}$ is profinite iff it is Hausdorff.
(b) $u\mathcal{M}$ is 0-dimensional iff it is profinite.

Proof. (a) (\Rightarrow) is trivial. (\Leftarrow) holds, as $u\mathcal{M}$ is Hausdorff iff $H(u\mathcal{M})$ is trivial.

(b) (\Leftarrow) is trivial. For (\Rightarrow) , since $H(u\mathcal{M}) = \bigcap_U \text{cl}_\tau(U)$ with the intersection taken over all τ -open neighbourhoods U of u , we get $H(u\mathcal{M}) \subseteq \bigcap_U U$, where the intersection is taken over all τ -clopen neighbourhoods of u . Since $u\mathcal{M}$ is T_1 and

0-dimensional, the previous intersection is just $\{u\}$, so $H(u\mathcal{M})$ is trivial. Hence, $u\mathcal{M} = u\mathcal{M}/H(u\mathcal{M})$ is a compact, Hausdorff, 0-dimensional topological group, so it is profinite. \square

A mapping $\Phi : X \rightarrow Y$ between two G -flows (G, X) and (G, Y) is a G -flow homomorphism if it is continuous and for every $g \in G$ and $x \in X$ we have $\Phi(gx) = g\Phi(x)$. A surjective [bijective] G -flow homomorphism is a G -flow epimorphism [G -flow isomorphism]; note that a G -flow epimorphism is necessarily a topological quotient map, and an inverse of a G -flow isomorphism is necessarily a G -flow isomorphism.

The proof of the next fact is basically the argument from the proof of [25, Proposition 5.41], so we will not repeat it.

Fact 2.3. *Let (G, X) and (G, Y) be two G -flows, and let $\Phi : \text{EL}(X) \rightarrow \text{EL}(Y)$ be a G -flow and semigroup epimorphism. Let \mathcal{M} be a minimal left ideal of $\text{EL}(X)$ and $u \in \mathcal{J}(\mathcal{M})$. Then:*

- (i) $\mathcal{M}' := \Phi[\mathcal{M}]$ is a minimal left ideal of $\text{EL}(Y)$ and $u' := \Phi(u) \in \mathcal{J}(\mathcal{M}')$;
- (ii) $\Phi|_{u\mathcal{M}} : u\mathcal{M} \rightarrow u'\mathcal{M}'$ is a group epimorphism and a quotient map in the τ -topologies. \square

Corollary 2.4. *Let (G, X) and (G, Y) be two G -flows, and let $\Phi : \text{EL}(X) \rightarrow \text{EL}(Y)$ be a G -flow and semigroup isomorphism. Let \mathcal{M} be a minimal left ideal of $\text{EL}(X)$ and $u \in \mathcal{J}(\mathcal{M})$. Then $\Phi|_{u\mathcal{M}} : u\mathcal{M} \rightarrow \Phi(u)\Phi[\mathcal{M}]$ is a group isomorphism and a homeomorphism (in the τ -topologies). \square*

A natural way of obtaining a G -flow and semigroup epimorphism $\text{EL}(X) \rightarrow \text{EL}(Y)$ is to induce it from a G -flow epimorphism $X \rightarrow Y$ or a G -flow monomorphism $Y \rightarrow X$. This is explained in the next fact whose proof is left as an exercise.

Fact 2.5. *Let (G, X) and (G, Y) be G -flows.*

- (i) *If $f : X \rightarrow Y$ is a G -flow epimorphism, then it induces a G -flow and semigroup epimorphism $\hat{f} : \text{EL}(X) \rightarrow \text{EL}(Y)$ given by:*

$$\hat{f}(\eta)(y) = f(\eta(x))$$

for $\eta \in \text{EL}(X)$ and $y \in Y$, where $x \in X$ is any element such that $f(x) = y$.

- (ii) *If $f : Y \rightarrow X$ is a G -flow monomorphism, then it induces a G -flow and semigroup epimorphism $\hat{f} : \text{EL}(X) \rightarrow \text{EL}(Y)$ given by:*

$$\hat{f}(\eta)(y) = f^{-1}(\eta(f(y)))$$

for $\eta \in \text{EL}(X)$ and $y \in Y$, where $f^{-1} : \text{Im}(f) \rightarrow Y$ is the inverse of f . \square

Corollary 2.6. *Let $f : X \rightarrow Y$ be a G -flow epimorphism and let $\hat{f} : \text{EL}(X) \rightarrow \text{EL}(Y)$ be the induced epimorphism given by Fact 2.5(i). Then, for any $\eta \in \text{EL}(X)$, $\text{Im}(\hat{f}(\eta)) = f[\text{Im}(\eta)]$. Thus, if $\text{Im}(\eta)$ is finite, so is $\text{Im}(\hat{f}(\eta))$. \square*

Corollary 2.7. *If (G, Y) is a subflow of (G, X) and there is an element $\eta \in \text{EL}(X)$ with $\text{Im}(\eta) \subseteq Y$, then the Ellis groups of the flows X and Y are topologically isomorphic.*

Proof. Let $f = \text{id} : Y \rightarrow X$. Then the map $\hat{f} : \text{EL}(X) \rightarrow \text{EL}(Y)$ from Fact 2.5(ii) is the restriction to Y . Let \mathcal{M} be a minimal left ideal of $\text{EL}(X)$. Replacing η by any element of $\eta\mathcal{M}$, we can assume that $\eta \in \mathcal{M}$. Let $u \in \mathcal{J}(\mathcal{M})$ satisfy $\eta \in u\mathcal{M}$. Then $\text{Im}(u) = \text{Im}(\eta) \subseteq Y$. Let $\mathcal{M}' = \hat{f}[\mathcal{M}]$ and $u' = \hat{f}(u)$. By Fact

2.3. $u'\mathcal{M}'$ is the Ellis group of (G, Y) and $\hat{f}_{\upharpoonright u\mathcal{M}} : u\mathcal{M} \rightarrow u'\mathcal{M}'$ is an epimorphism and a topological quotient map. Finally, $\hat{f}_{\upharpoonright u\mathcal{M}}$ is injective, as for $\tau_1, \tau_2 \in u\mathcal{M}$: $\tau_1|_Y = \tau_2|_Y \implies \tau_1|_{\text{Im}(u)} = \tau_2|_{\text{Im}(u)} \iff \tau_1 u = \tau_2 u \iff \tau_1 = \tau_2$. \square

We further consider an inverse system of G -flows $((G, X_i))_{i \in I}$, where I is a directed set, and for each $i < j$ in I a G -flow epimorphism $\pi_{i,j} : X_j \rightarrow X_i$ is given. Then $X := \varprojlim_{i \in I} X_i$ is a compact, Hausdorff space and G acts naturally and continuously on X . Denote by $\pi_i : X \rightarrow X_i$ the natural projections; they are G -flow epimorphisms. By Fact 2.5, we have G -flow and semigroup epimorphisms $\hat{\pi}_{i,j} : \text{EL}(X_j) \rightarrow \text{EL}(X_i)$ for $i < j$ and $\hat{\pi}_i : \text{EL}(X) \rightarrow \text{EL}(X_i)$. It turns out that the previous inverse limit construction transfers to Ellis semigroups, and furthermore to Ellis groups. We state this in the next fact; for the proof see [25, Lemma 6.42].

Fact 2.8. *Fix the notation from the previous paragraph.*

The family $(\text{EL}(X_i))_{i \in I}$, together with the mappings $\hat{\pi}_{i,j}$, is an inverse system of semigroups and of G -flows. There exists a natural G -flow and semigroup isomorphism $\text{EL}(X) \cong \varprojlim_{i \in I} \text{EL}(X_i)$, and after identifying $\text{EL}(X)$ with $\varprojlim_{i \in I} \text{EL}(X_i)$, the natural projections of this inverse limit are just the $\hat{\pi}_i$'s.

For every minimal left ideal \mathcal{M} of $\text{EL}(X)$, each $\mathcal{M}_i := \hat{\pi}_i[\mathcal{M}]$ is a minimal left ideal of $\text{EL}(X_i)$ and $\mathcal{M} = \varprojlim_{i \in I} \mathcal{M}_i$. Also, if $u \in \mathcal{J}(\mathcal{M})$, then $u_i := \hat{\pi}_i(u) \in \mathcal{J}(\mathcal{M}_i)$. Furthermore, $u\mathcal{M} = \varprojlim_{i \in I} u_i \mathcal{M}_i$ and the τ -topology on $u\mathcal{M}$ coincides with the inverse limit topology induced from the τ -topologies on the $u_i \mathcal{M}_i$'s. \square

The material in the rest of this subsection will be needed in the analysis of Examples 5.10 and 5.11.

Recall that a G -ambit is a G -flow (G, X, x_0) with a distinguished point x_0 with dense orbit. It is well-known that for any discrete group G , $(G, \beta G, \mathcal{U}_e)$ is the universal G -ambit, where \mathcal{U}_e is the principal ultrafilter concentrated on e ; we will identify e with \mathcal{U}_e . This easily yields a unique left continuous semigroup operation on βG extending the action of G on βG , and for any G -flow (G, X) this also yields a unique left continuous action $*$ of the semigroup βG on (G, X) extending the action of G . Thus, there is a unique G -flow and semigroup epimorphism $\beta G \rightarrow \text{EL}(X)$ (mapping e to the identity): it is given by $p \mapsto l_p$, where $l_p(x) := p * x$. Applying this to $X := \beta G$, we get an isomorphism $\beta G \cong \text{EL}(\beta G)$. For some details on this see [7, Chapter 1].

It is also well-known (see [8, Exercise 1.25]) that for the Bernoulli shift $X := 2^G$ the above map $\beta G \rightarrow \text{EL}(X)$ is an isomorphism of flows and of semigroups, i.e. $\beta G \cong \text{EL}(2^G)$.

Proposition 2.9. *Let (G, X) be a G -flow for which there is a G -flow epimorphism $\varphi : X \rightarrow 2^G$. Then,*

- (i) $\text{EL}(X) \cong \beta G$ as G -flows and as semigroups.
- (ii) *The Ellis group of X is topologically isomorphic with the Ellis group of βG .*

Proof. (i) By Fact 2.5(i), we have the induced G -flow and semigroup epimorphism $\hat{\varphi} : \text{EL}(X) \rightarrow \text{EL}(2^G)$. On the other hand, by the above discussion, there is a G -flow and semigroup epimorphism $\psi : \beta G \rightarrow \text{EL}(X)$. So the composition $\hat{\varphi} \circ \psi : \beta G \rightarrow \text{EL}(2^G)$ is a G -flow and semigroup epimorphism. By the above discussion, it is an isomorphism. Therefore, ψ must be an isomorphism, too. (ii) follows from (i) and Corollary 2.4. \square

Recall that the Bohr compactification of a topological group G is a unique (up to isomorphism) universal object in the category of group compactifications of G , i.e. continuous homomorphisms $G \rightarrow H$ with dense image, where H is a compact, Hausdorff group. We often identify the Bohr compactification with the target compact group, and denote it by bG . For a discrete group G , a minimal left ideal \mathcal{M} in βG and an idempotent $u \in \mathcal{M}$, the quotient $u\mathcal{M}/H(u\mathcal{M})$ turns out to be the generalized Bohr compactification of G in the terminology from [7]. There is always a continuous epimorphism from the generalized Bohr compactification to the Bohr compactification. We will need the following result, which is Corollary 4.3 of [7]. (See also [16] for a more general, model-theoretic account.) We do not recall here the notion of strongly amenable group. We only need to know that abelian groups are strongly amenable. For more details see [7].

Fact 2.10. *If a discrete group G is strongly amenable (e.g. G is abelian), then the generalized Bohr compactification of G and the Bohr compactification of G coincide. \square*

We will also need the following consequence of the presentation of bG (for an abelian discrete group G) as the “double Pontryagin dual”, which can be found in [29]; a short proof based on Pontryagin duality is given in [5, Section 1].

Fact 2.11. *Let G be a discrete abelian group. Then, bG is profinite iff G is of finite exponent. \square*

2.3. Structural Ramsey theory. Recall that a first order structure K is called a *Fraïssé structure* if it is countable, locally finite (finitely generated substructures of K are finite; although this property is not always taken as a part of the definition) and ultrahomogeneous (every isomorphism between finite substructures of K lifts to an automorphism of K). Every Fraïssé structure K is uniquely determined by its *age*, $\text{Age}(K)$, i.e. the class of all finite structures which are embeddable in K . The age of a Fraïssé structure is a *Fraïssé class*, i.e. a class of finite structures which is closed under isomorphisms, countable up to isomorphism, and satisfies *hereditary*, *joint embedding* and *amalgamation property*. On the other hand, for every Fraïssé class \mathcal{C} of first order structures there exists a unique Fraïssé structure, called the *Fraïssé limit* of \mathcal{C} , whose age is exactly \mathcal{C} .

Structural Ramsey theory, invented by Nešetřil and Rödl in the 1970s, investigates combinatorial properties of Fraïssé classes (and, more generally, categories). Originally, these structural combinatorial properties are given by colorings of isomorphic copies of A in B , where A and B are members of the given class such that A is embeddable in B . We follow the approach from [31]. Let K be a Fraïssé structure. For $A, B \in \text{Age}(K)$, $\text{Emb}(A, B)$ stands for the set of all embeddings $A \rightarrow B$. K (or $\text{Age}(K)$) has *separately finite embedding Ramsey degree* if for every $A \in \text{Age}(K)$ there exists $l < \omega$ such that for every $B \in \text{Age}(K)$ with $\text{Emb}(A, B) \neq \emptyset$ and every $r < \omega$ there exists $C \in \text{Age}(K)$ such that for every coloring $c : \text{Emb}(A, C) \rightarrow r$ there exists $f \in \text{Emb}(B, C)$ with $\#c[f \circ \text{Emb}(A, B)] \leq l$. We added the word “separately” to emphasize that l depends on A . If in the previous definition l can be chosen to be 1 for every A , we obtain the notion of the *embedding Ramsey property*. By [31, Proposition 4.4], K has separately finite embedding Ramsey degree iff it has separately finite structural Ramsey degree (defined by using colorings of isomorphic copies of A in B in place of embeddings). Moreover, by [31, Corollary 4.5], K has

the embedding Ramsey property iff it has the structural Ramsey property and all structures in $\text{Age}(K)$ are rigid (have trivial automorphism groups).

Both the property of having separately finite embedding Ramsey degree and the embedding Ramsey property for a Fraïssé structure K can be alternatively defined as follows. For finite $\bar{a} \subseteq K$ and $C \subseteq K$ denote by $\binom{C}{\bar{a}}^{\text{qf}}$ the set of all $\bar{a}' \subseteq C$ such that $\bar{a}' \equiv^{\text{qf}} \bar{a}$ (\bar{a}' and \bar{a} have the same quantifier-free type).

Fact 2.12. *A Fraïssé structure K has separately finite embedding Ramsey degree iff for every finite $\bar{a} \subseteq K$ there exists $l < \omega$ such that for any finite $\bar{b} \subseteq K$ containing \bar{a} and $r < \omega$ there exists a finite $C \subseteq K$ such that for every coloring $c : \binom{C}{\bar{a}}^{\text{qf}} \rightarrow r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}^{\text{qf}}$ with $\#c[\binom{\bar{b}'}{\bar{a}}^{\text{qf}}] \leq l$.*

The same holds for the embedding Ramsey property and $l = 1$. \square

2.4. Flows in model theory. As mentioned in the introduction, the methods of topological dynamics were introduced to model theory by Newelski over ten years ago, and since then this approach has gained a lot of attention which resulted in various applications (e.g. to strong types in [18, 19]). Flows in model theory occur naturally in two ways. One way is to consider the action of a definable group on various spaces of types concentrated on this group. The other is to consider the action of the automorphism group of a model on various spaces of types. In this paper, we consider the second situation. We are mostly interested in the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where \mathfrak{C} is a monster model of a theory T , \bar{c} is an enumeration of \mathfrak{C} , $S_{\bar{c}}(\mathfrak{C})$ is the space of all types over \mathfrak{C} extending $\text{tp}(\bar{c})$, and the action of $\text{Aut}(\mathfrak{C})$ on $S_{\bar{c}}(\mathfrak{C})$ is the obvious one.

Recall that the question when the group $\text{Gal}_{\text{KP}}(T)$ is profinite was an initial motivation behind this paper. In [18] it is proved that there is a sequence of quotient topological maps and group epimorphisms:

$$u\mathcal{M} \rightarrow u\mathcal{M}/H(u\mathcal{M}) \rightarrow \text{Gal}_{\text{L}}(T) \rightarrow \text{Gal}_{\text{KP}}(T),$$

for any minimal left ideal \mathcal{M} of $\text{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $u \in \mathcal{J}(\mathcal{M})$, where $u\mathcal{M}$ is equipped with the τ -topology, $u\mathcal{M}/H(u\mathcal{M})$ with the corresponding quotient topology, and $\text{Gal}_{\text{L}}(T)$ and $\text{Gal}_{\text{KP}}(T)$ with the topologies described in Subsection 2.1. The following easy consequence of this result gives a sufficient condition for profiniteness of $\text{Gal}_{\text{KP}}(T)$.

Proposition 2.13. *$\text{Gal}_{\text{KP}}(T)$ is profinite if $u\mathcal{M}/H(u\mathcal{M})$ is profinite. Furthermore, $u\mathcal{M}/H(u\mathcal{M})$ is profinite if $u\mathcal{M}$ is 0-dimensional.*

Proof. Both assertions are consequences of the following more general, easy fact: If G is a compact semitopological group, H is a Hausdorff topological group, and $f : G \rightarrow H$ is a quotient topological map and group epimorphism, then H is profinite if G is 0-dimensional. \square

Thus, by Remark 2.2, for profiniteness of $\text{Gal}_{\text{KP}}(T)$ it is enough to prove profiniteness (equiv. 0-dimensionality) of either $u\mathcal{M}$ or $u\mathcal{M}/H(u\mathcal{M})$.

The investigation of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ in concrete examples is not easy, thus we need reductions of it to flows which are easier to deal with. Some reductions of this kind appeared in Section 2 of [15], but here we need more specific ones.

Let \bar{d} be a tuple of all elements of \mathfrak{C} in which each element of \mathfrak{C} is repeated infinitely many times. It is clear that $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{c}}(\mathfrak{C})$ are isomorphic $\text{Aut}(\mathfrak{C})$ -flows.

Hence, their Ellis semigroups are isomorphic as semigroups and as $\text{Aut}(\mathfrak{C})$ -flows. So, in fact, one can work with $S_{\bar{d}}(\mathfrak{C})$ in place of $S_{\bar{e}}(\mathfrak{C})$ whenever it is convenient.

Let us fix some notation. Let \bar{x} be a tuple of variables corresponding to \bar{d} . For $\bar{x}' \subseteq \bar{x}$, \bar{z} with $|\bar{z}| = |\bar{x}'|$, and for $p(\bar{x}') \in S_{\bar{x}'}(\mathfrak{C})$ we denote by $p[\bar{x}'/\bar{z}]$ the type from $S_{\bar{z}}(\mathfrak{C})$ obtained by replacing the variables \bar{x}' by \bar{z} . Denote by \bar{z} a sequence $(z_i)_{i < n}$ of variables, where $n \leq \omega$. We discuss connections between the $\text{Aut}(\mathfrak{C})$ -flows $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{z}}(\mathfrak{C})$. The proof of the next lemma is left as an exercise.

Lemma 2.14. *Let $\Phi : \text{EL}(S_{\bar{d}}(\mathfrak{C})) \rightarrow \text{EL}(S_{\bar{z}}(\mathfrak{C}))$ be defined by $\Phi(\eta) := \hat{\eta}$, where $\hat{\eta}$ is given by:*

$$\hat{\eta}(p(\bar{z})) = \eta(q(\bar{x}))_{\upharpoonright_{\bar{x}'}}[\bar{x}'/\bar{z}],$$

where $p(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$, and $\bar{x}' \subseteq \bar{x}$ and $q(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ are such that $q(\bar{x})_{\upharpoonright_{\bar{x}'}}[\bar{x}'/\bar{z}] = p(\bar{z})$. Then Φ is a well-defined epimorphism of $\text{Aut}(\mathfrak{C})$ -flows and of semigroups. \square

Having in mind that the flows $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{e}}(\mathfrak{C})$ are isomorphic, we get:

Corollary 2.15. *There exists an $\text{Aut}(\mathfrak{C})$ -flow and semigroup epimorphism from $\text{EL}(S_{\bar{e}}(\mathfrak{C}))$ to $\text{EL}(S_{\bar{z}}(\mathfrak{C}))$. In particular, such an epimorphism exists from $\text{EL}(S_{\bar{e}}(\mathfrak{C}))$ to $\text{EL}(S_n(\mathfrak{C}))$ for all $n < \omega$. \square*

Corollary 2.16. *If $\bar{z} = (z_i)_{i < \omega}$, then $\text{EL}(S_{\bar{e}}(\mathfrak{C}))$ and $\text{EL}(S_{\bar{z}}(\mathfrak{C}))$ are $\text{Aut}(\mathfrak{C})$ -flow and semigroup isomorphic. Thus, the Ellis groups of the flows $(\text{Aut}(\mathfrak{C}), S_{\bar{e}}(\mathfrak{C}))$ and $(\text{Aut}(\mathfrak{C}), S_{\bar{z}}(\mathfrak{C}))$ are topologically isomorphic.*

Proof. It is enough to prove that Φ from Lemma 2.14 is injective. Let $\eta_1 \neq \eta_2 \in \text{EL}(S_{\bar{d}}(\mathfrak{C}))$, and take $q(\bar{x})$ and $\varphi(\bar{x}, \bar{a})$ such that $\varphi(\bar{x}, \bar{a}) \in \eta_1(q(\bar{x}))$ and $\neg\varphi(\bar{x}, \bar{a}) \in \eta_2(q(\bar{x}))$. Take $\bar{x}' \subseteq \bar{x}$ such that $|\bar{x}'| = \omega$ and all variables occurring in $\varphi(\bar{x}, \bar{a})$ are in \bar{x}' , so we may write $\varphi(\bar{x}, \bar{a})$ as $\varphi'(\bar{x}', \bar{a})$. Let $p(\bar{z}) = q(\bar{x})_{\upharpoonright_{\bar{x}'}}[\bar{x}'/\bar{z}]$. By the definition of $\hat{\eta}_1$ and $\hat{\eta}_2$, $\varphi'(\bar{z}, \bar{a}) \in \hat{\eta}_1(p(\bar{z}))$ and $\neg\varphi'(\bar{z}, \bar{a}) \in \hat{\eta}_2(p(\bar{z}))$. Thus, $\hat{\eta}_1 \neq \hat{\eta}_2$. \square

If $|\bar{z}| = n < \omega$, then in general we do not have injectivity of Φ , but we may distinguish a sufficient condition on T for which Φ is injective for some n . We say that a theory T is m -ary (for some $m < \omega$) if every L -formula is equivalent modulo T to a Boolean combination of L -formulae with at most m free variables.

Corollary 2.17. *If T is $(m+1)$ -ary, then $\text{EL}(S_{\bar{e}}(\mathfrak{C}))$ and $\text{EL}(S_m(\mathfrak{C}))$ are $\text{Aut}(\mathfrak{C})$ -flow and semigroup isomorphic. Thus, the Ellis groups of the flows $(\text{Aut}(\mathfrak{C}), S_{\bar{e}}(\mathfrak{C}))$ and $(\text{Aut}(\mathfrak{C}), S_m(\mathfrak{C}))$ are topologically isomorphic.*

Proof. If $|\bar{z}| = m$, we prove that Φ is injective. Let $\eta_1 \neq \eta_2 \in \text{EL}(S_{\bar{d}}(\mathfrak{C}))$ and let $q(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ be such that $\eta_1(q(\bar{x})) \neq \eta_2(q(\bar{x}))$. By $(m+1)$ -arity of the theory, there is a formula $\varphi(\bar{x}'', \bar{y})$ with $\bar{x}'' \subseteq \bar{x}$ and $|\bar{x}''| + |\bar{y}| \leq m+1$, and \bar{a} such that $\varphi(\bar{x}'', \bar{a}) \in \eta_1(q(\bar{x}))$ and $\neg\varphi(\bar{x}'', \bar{a}) \in \eta_2(q(\bar{x}))$. Note that $|\bar{y}| \geq 1$, as otherwise $\varphi(\bar{x}'', \bar{a})$ is a formula without parameters, so it belongs to $\eta_1(q(\bar{x}))$ iff it belongs to $\eta_2(q(\bar{x}))$. So $|\bar{x}''| \leq m$, and we can write $\varphi(\bar{x}'', \bar{a})$ as $\varphi'(\bar{x}', \bar{a})$ for some $\bar{x}' \subseteq \bar{x}$ such that $|\bar{x}'| = m$. Now, the final step of the proof of Corollary 2.16 goes through. \square

We will need the following general, easy observation.

Remark 2.18. Let (G, X) and (G, Y) be G -flows. Denote by $\text{Inv}(X)$ and $\text{Inv}(Y)$ the sets of all points fixed by G in X and Y , respectively. Assume that $F : \text{EL}(X) \rightarrow \text{EL}(Y)$ is a G -flow epimorphism. Then for any $\eta \in \text{EL}(X)$ we have: $\text{Im}(\eta) \subseteq \text{Inv}(X)$ iff $\text{Im}(\eta) = \text{Inv}(X)$, and these equivalent conditions imply $\text{Im}(F(\eta)) = \text{Inv}(Y)$. So, if F is an isomorphism, then $\text{Im}(\eta) = \text{Inv}(X)$ iff $\text{Im}(F(\eta)) = \text{Inv}(Y)$. \square

By Corollary 2.17 and Remark 2.18, we get:

Corollary 2.19. *If T is $(m + 1)$ -ary, then the existence of $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(\eta) = \text{Inv}_{\bar{c}}(\mathfrak{C})$ is equivalent to the existence of $\eta' \in \text{EL}(S_m(\mathfrak{C}))$ with $\text{Im}(\eta') = \text{Inv}_m(\mathfrak{C})$ (where $\text{Inv}_m(\mathfrak{C})$ is the set of all invariant types in $S_m(\mathfrak{C})$). \square*

We will need one more consequence of the above investigations. Let $L \subseteq L^*$ be two languages, let T^* be a complete L^* -theory and $T := T^*_{\uparrow L}$. Assume that \mathfrak{C}^* is a monster of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster of T . Let \bar{c} be an enumeration of \mathfrak{C} (and \mathfrak{C}^*). We can treat $S_{\bar{c}}(\mathfrak{C})$ as an $\text{Aut}(\mathfrak{C}^*)$ -flow, and when we do that (in this section) we write $S_{\bar{c}}^*(\mathfrak{C})$ in place of $S_{\bar{c}}(\mathfrak{C})$; similarly for $S_{\bar{z}}^*(\mathfrak{C})$. Note that $\text{Aut}(\mathfrak{C}^*) \leq \text{Aut}(\mathfrak{C})$, and so $\text{EL}(S_{\bar{c}}^*(\mathfrak{C})) \subseteq \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $\text{EL}(S_{\bar{z}}^*(\mathfrak{C})) \subseteq \text{EL}(S_{\bar{z}}(\mathfrak{C}))$.

Lemma 2.20. *Take the previous notation.*

- (i) *There is an $\text{Aut}(\mathfrak{C}^*)$ -flow and semigroup epimorphism $\Psi : \text{EL}(S_{\bar{c}}(\mathfrak{C}^*)) \rightarrow \text{EL}(S_{\bar{c}}^*(\mathfrak{C}))$.*
- (ii) *If there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$ (equiv. = $\text{Inv}_{\bar{c}}(\mathfrak{C}^*)$), then there is $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta) = \text{Inv}_{\bar{c}}^*(\mathfrak{C})$, where $\text{Inv}_{\bar{c}}(\mathfrak{C}^*) \subseteq S_{\bar{c}}(\mathfrak{C}^*)$ and $\text{Inv}_{\bar{c}}^*(\mathfrak{C}) \subseteq S_{\bar{c}}^*(\mathfrak{C})$ are the subsets of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types in $S_{\bar{c}}(\mathfrak{C}^*)$ and $S_{\bar{c}}^*(\mathfrak{C})$, respectively.*
- (iii) *There is an $\text{Aut}(\mathfrak{C}^*)$ -flow and semigroup epimorphism $\Psi_{\bar{z}} : \text{EL}(S_{\bar{c}}(\mathfrak{C}^*)) \rightarrow \text{EL}(S_{\bar{z}}^*(\mathfrak{C}))$.*
- (iv) *If there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$ (equiv. = $\text{Inv}_{\bar{c}}(\mathfrak{C}^*)$), then there is $\eta \in \text{EL}(S_{\bar{z}}(\mathfrak{C}))$ such that $\text{Im}(\eta) = \text{Inv}_{\bar{z}}^*(\mathfrak{C})$, where $\text{Inv}_{\bar{z}}(\mathfrak{C}^*) \subseteq S_{\bar{z}}^*(\mathfrak{C})$ is the subset of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types in $S_{\bar{z}}^*(\mathfrak{C})$.*

Proof. (i) Let $\bar{z} = (z_i)_{i < \omega}$. By Corollary 2.16, we have an $\text{Aut}(\mathfrak{C}^*)$ -flow and semigroup isomorphism $\Phi^* : \text{EL}(S_{\bar{c}}(\mathfrak{C}^*)) \rightarrow \text{EL}(S_{\bar{z}}(\mathfrak{C}^*))$, and an $\text{Aut}(\mathfrak{C})$ -flow and semigroup isomorphism $\Phi : \text{EL}(S_{\bar{c}}(\mathfrak{C})) \rightarrow \text{EL}(S_{\bar{z}}(\mathfrak{C}))$. Since $\text{Aut}(\mathfrak{C}^*) \leq \text{Aut}(\mathfrak{C})$, we get that $\Phi_{\uparrow \text{EL}(S_{\bar{c}}^*(\mathfrak{C}))}$ is an $\text{Aut}(\mathfrak{C}^*)$ -flow and semigroup isomorphism $\text{EL}(S_{\bar{c}}^*(\mathfrak{C})) \rightarrow \text{EL}(S_{\bar{z}}^*(\mathfrak{C}))$. We also have an $\text{Aut}(\mathfrak{C}^*)$ -flow epimorphism $S_{\bar{z}}(\mathfrak{C}^*) \rightarrow S_{\bar{z}}^*(\mathfrak{C})$ given by the restriction of the language; hence, by Fact 2.5, there exists an $\text{Aut}(\mathfrak{C}^*)$ -flow and semigroup epimorphism $\Theta : \text{EL}(S_{\bar{z}}(\mathfrak{C}^*)) \rightarrow \text{EL}(S_{\bar{z}}^*(\mathfrak{C}))$. Then $\Psi := \Phi_{\uparrow \text{EL}(S_{\bar{c}}^*(\mathfrak{C}))}^{-1} \circ \Theta \circ \Phi^*$ is as required.

(ii) Since $\text{EL}(S_{\bar{c}}^*(\mathfrak{C})) \subseteq \text{EL}(S_{\bar{c}}(\mathfrak{C}))$, (ii) follows from (i) and Remark 2.18 applied to $X := S_{\bar{c}}(\mathfrak{C}^*)$, $Y := S_{\bar{c}}^*(\mathfrak{C})$, and $G := \text{Aut}(\mathfrak{C}^*)$.

(iii) $\Psi_{\bar{z}} := \Theta \circ \Phi^*$ from the proof of (i) does the job (but here $\Phi^* : \text{EL}(S_{\bar{c}}(\mathfrak{C}^*)) \rightarrow \text{EL}(S_{\bar{z}}(\mathfrak{C}^*))$ is an epimorphism provided by Corollary 2.15).

(iv) follows from (iii) and Remark 2.18. \square

We now describe a natural way of presenting $S_{\bar{c}}(\mathfrak{C})$ as an inverse limit of $\text{Aut}(\mathfrak{C})$ -flows, which is one of the key tools in this paper.

For any $\bar{a} \subseteq \mathfrak{C}^* \succ \mathfrak{C}$, $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ where $|\bar{x}| = |\bar{a}|$, and $\bar{p} = \{p_j\}_{j < m} \subseteq S_{\bar{y}}(T)$, by $\text{tp}_{\Delta}(\bar{a}/\bar{p})$ we mean the Δ -type of \bar{a} over $\bigcup_{j < m} p_j(\mathfrak{C})$, i.e. the set of all formulae of the form $\varphi_i(\bar{x}, \bar{b})^\epsilon$ such that $\epsilon \in 2$, \bar{b} realizes one of p_j 's, and $\models \varphi_i(\bar{a}, \bar{b})^\epsilon$. (Here, as usual, φ^0 denotes $\neg\varphi$, and φ^1 denotes φ .)

For $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$, where \bar{x} is reserved for \bar{c} , and $\bar{p} = \{p_j\}_{j < m} \subseteq S_{\bar{y}}(T)$, by $S_{\bar{c}, \Delta}(\bar{p})$ we denote the space of all complete Δ -types over $\bigcup_{j < m} p_j(\mathfrak{C})$ consistent with $\text{tp}(\bar{c})$; equivalently:

$$S_{\bar{c}, \Delta}(\bar{p}) = \{\text{tp}_{\Delta}(\bar{c}^*/\bar{p}) \mid \bar{c}^* \subseteq \mathfrak{C}^* \text{ and } \bar{c}^* \equiv \bar{c}\}.$$

In the usual way, we endow $S_{\bar{c}, \Delta}(\bar{p})$ with a topology, turning it into a 0-dimensional, compact, Hausdorff space. Moreover, it is naturally an $\text{Aut}(\mathfrak{C})$ -flow.

Let \mathcal{F} be the family of all pairs $(\Delta = \{\varphi_i\}_{i < k}, \bar{p} = \{p_j\}_{j < m})$ as above. We order \mathcal{F} naturally by:

$$(\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}, \bar{p} = \{p_j\}_{j < m}) \leq (\Delta' = \{\varphi'_i(\bar{x}, \bar{y}')\}_{i < k'}, \bar{p}' = \{p'_j\}_{j < m'})$$

if $\bar{y} \subseteq \bar{y}'$, $\{\varphi_i(\bar{x}, \bar{y})\}_{i < k} \subseteq \{\varphi'_i(\bar{x}, \bar{y}')\}_{i < k'}$ by using dummy variables, and $\{p_j\}_{j < m} \subseteq \{p'_j\}_{j < m'}$. Then \mathcal{F} is directed by \leq , and for pairs $t = (\Delta, \bar{p})$ and $t' = (\Delta', \bar{p}')$ in \mathcal{F} , if $t \leq t'$, we have an $\text{Aut}(\mathfrak{C})$ -flow epimorphism given by the restriction:

$$\pi_{t, t'} : S_{\bar{c}, \Delta'}(\bar{p}') \rightarrow S_{\bar{c}, \Delta}(\bar{p}).$$

Therefore, we have an inverse system of $\text{Aut}(\mathfrak{C})$ -flows $((\text{Aut}(\mathfrak{C}), S_{\bar{c}, \Delta}(\bar{p}))_{(\Delta, \bar{p}) \in \mathcal{F}}$, and we clearly have:

Lemma 2.21. $S_{\bar{c}}(\mathfrak{C}) \cong \varprojlim_{(\Delta, \bar{p}) \in \mathcal{F}} S_{\bar{c}, \Delta}(\bar{p})$ as $\text{Aut}(\mathfrak{C})$ -flows. \square

The usual $\text{Aut}(\mathfrak{C})$ -flow $S_{\bar{c}, \Delta}(\mathfrak{C})$ of complete Δ -types over the whole \mathfrak{C} consistent with $\text{tp}(\bar{c})$ projects onto $S_{\bar{c}, \Delta}(\bar{p})$. We also have $S_{\bar{c}}(\mathfrak{C}) \cong \varprojlim_{\Delta \in \mathcal{F}} S_{\bar{c}, \Delta}(\mathfrak{C})$, but this presentation is not sufficient to be used in our main results (see Example 5.9).

2.5. Contents and strong heirs. The following definition was introduced in [15].

Definition 2.22 ([15, Definition 3.1]). Fix $A \subseteq B$.

(a) For $p(\bar{x}) \in S(B)$, the *content* of p over A is defined as:

$$\text{ct}_A(p) = \{(\varphi(\bar{x}, \bar{y}), q(\bar{y})) \in L(A) \times S(A) \mid \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \text{ for some } \bar{b} \models q\}.$$

(b) The content of a sequence $p_0(\bar{x}), \dots, p_{n-1}(\bar{x}) \in S(B)$ over A , $\text{ct}_A(p_0, \dots, p_{n-1})$, is defined as the set of all $(\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y}), q(\bar{y})) \in L(A)^n \times S(A)$ such that for some $\bar{b} \models q$ and for every $i < n$ we have $\varphi_i(\bar{x}, \bar{b}) \in p_i$.

If $A = \emptyset$, we write just $\text{ct}(p)$ and $\text{ct}(p_0, \dots, p_{n-1})$.

A fundamental connection between contents and the Ellis semigroup is given by:

Fact 2.23 ([15, Proposition 3.5]). Let $\pi(\bar{x})$ be a type over \emptyset , and (p_0, \dots, p_{n-1}) and (q_0, \dots, q_{n-1}) sequences from $S_\pi(\mathfrak{C})$. Then $\text{ct}(q_0, \dots, q_{n-1}) \subseteq \text{ct}(p_0, \dots, p_{n-1})$ iff there exists $\eta \in \text{EL}(S_\pi(\mathfrak{C}))$ such that $\eta(p_i) = q_i$ for every $i < n$. \square

As was explained in [15], contents expand the concept of fundamental order of Lascar and Poizat, hence an analogous notion of heir can be defined.

Definition 2.24 ([15, Definition 3.2]). Let $M \subseteq A$ and $p(\bar{x}) \in S(A)$. $p(\bar{x})$ is a *strong heir* over M if for every finite $\bar{m} \subseteq M$ and $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$, where $\bar{a} \subseteq A$ is finite and $\varphi(\bar{x}, \bar{y}) \in L(M)$, there is $\bar{a}' \subseteq M$ such that $\varphi(\bar{x}, \bar{a}') \in p(\bar{x})$ and $\text{tp}(\bar{a}'/\bar{m}) = \text{tp}(\bar{a}/\bar{m})$.

Fact 2.25 ([15, Lemma 3.3]). Let $M \subseteq A$, where M is \aleph_0 -saturated. Then every $p(\bar{x}) \in S(M)$ has an extension $p'(\bar{x}) \in S(A)$ which is a strong heir over M . \square

2.6. Amenability of a theory. Amenable and extremely amenable theories were introduced and studied in [12]. We will not give the original definitions but rather the characterizations which we will use in this paper. For the details see [12].

Definition 2.26. (a) A theory T is *amenable* if every finitary type $p \in S(T)$ is *amenable*, i.e. there exists an invariant, (regular) Borel probability measure on $S_p(\mathfrak{C})$.

- (b) A theory T is *extremely amenably* if every finitary type $p \in S(T)$ is *extremely amenable*, i.e. there exists an invariant type in $S_p(\mathfrak{C})$.

In fact, in the above definitions we can remove the adjective ‘‘finitary’’. We also get the same notions if we use only $p := \text{tp}(\bar{c})$ (where \bar{c} is an enumeration of \mathfrak{C}).

These definitions do not depend on the choice of the monster model \mathfrak{C} , i.e. they are indeed properties of the theory T . In fact, it is enough to assume only that \mathfrak{C} is \aleph_0 -saturated and strongly \aleph_0 -homogeneous.

One should also recall that a regular, Borel probability measure on the space $S_p(\mathfrak{C})$ (or on any 0-dimensional compact space) is the same thing as a Keisler measure, i.e. finitely additive probability measure on the Boolean algebra of all clopen sets. All such measures form a compact subspace \mathfrak{M}_p of $[0, 1]^{\text{clopens}}$ equipped with the product topology.

3. DEFINABLE STRUCTURAL RAMSEY THEORY AND TOPOLOGICAL DYNAMICS

As mentioned in the introduction, in order to find interactions between Ramsey-like properties of a given theory T and dynamical properties of T , one has to impose the appropriate definability conditions on colorings.

For a finite tuple \bar{a} and a subset $C \subseteq \mathfrak{C}$, by $\binom{C}{\bar{a}}$ we denote the set of all realizations of $\text{tp}(\bar{a})$ in C : $\binom{C}{\bar{a}} := \{\bar{a}' \in C^{|\bar{a}|} \mid \bar{a}' \equiv \bar{a}\}$. If instead of C we have a tuple \bar{d} , the meaning of $\binom{\bar{d}}{\bar{a}}$ is the same, i.e. it is $\binom{D}{\bar{a}}$, where D is the set of all coordinates of \bar{d} .

For $r < \omega$, a *coloring* of the realizations of $\text{tp}(\bar{a})$ in C into r colors is any mapping $c : \binom{C}{\bar{a}} \rightarrow r$. A subset $S \subseteq \binom{C}{\bar{a}}$ is *monochromatic* with respect to c if $c[S]$ is a singleton.

Definition 3.1. (a) A coloring $c : \binom{C}{\bar{a}} \rightarrow 2^n$ is *definable* if there are formulae with parameters $\varphi_i(\bar{x})$, $i < n$, such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \models \varphi_i(\bar{a}') \\ 0, & \models \neg\varphi_i(\bar{a}') \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

- (b) A coloring $c : \binom{C}{\bar{a}} \rightarrow 2^n$ is *externally definable* if there are formulae without parameters $\varphi_i(\bar{x}, \bar{y})$ and types $p_i(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$, $i < n$, such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \\ 0, & \neg\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

- (c) If Δ is a set of formulae in variables \bar{x} and \bar{y} and $q \in S_{\bar{y}}(\emptyset)$, then an externally definable coloring c is called *externally definable* (Δ, q) -*coloring* if all the formulae $\varphi_i(\bar{x}, \bar{y})$'s defining c are taken from Δ and $p_i(\bar{y}) \in S_q(\mathfrak{C})$ for $i < n$.

Remark 3.2. A coloring $c : \binom{C}{\bar{a}} \rightarrow 2^n$ is definable iff it is externally definable via realized (in \mathfrak{C}) types $p_0(\bar{y}), \dots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$. \square

Remark 3.3. (i) An externally definable coloring $c : \binom{C}{\bar{a}} \rightarrow 2^n$ given by $\varphi_i(\bar{x}, \bar{y})$ and $p_i(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$, $i < n$, can be defined by using n formulae $\psi_i(\bar{x}, \bar{z})$, $i < n$, and only one type $p(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$. Moreover, we can require that $|\bar{z}| = |\bar{c}|$ and $p(\bar{z}) \in S_{\bar{c}}(\mathfrak{C})$.

- (ii) In the definition of definable coloring, we can assume that all formulae $\varphi_i(\bar{x})$, $i < n$, have the same parameters \bar{e} , and then the coloring is externally definable witnessed by the single realized type $p(\bar{y}) := \text{tp}(\bar{e}/\mathfrak{C})$.
- (iii) In the definition of externally definable (Δ, q) -coloring, without loss of generality we can assume that $|\bar{y}| = |\bar{c}|$ and $q(\bar{y}) = \text{tp}(\bar{c}/\emptyset)$. Whenever $|\bar{y}| = |\bar{c}|$ and $q(\bar{y}) = \text{tp}(\bar{c}/\emptyset)$, instead of “externally definable (Δ, q) -coloring” we will just say *externally definable Δ -coloring*.

Proof. (i) Let $\bar{z} = (\bar{y}_0, \dots, \bar{y}_{n-1})$, where each \bar{y}_i is of length $|\bar{y}|$ (where wlog \bar{y} is finite) and the \bar{y}_i 's are pairwise disjoint. Let $p(\bar{z}) \in S_{\bar{z}}(\emptyset)$ be any completion of $\bigcup_{i < n} p_i(\bar{y}_i)$ and let $\psi_i(\bar{x}, \bar{z}) := \varphi_i(\bar{x}, \bar{y}_i)$. Then $\psi_i(\bar{a}', \bar{z}) \in p(\bar{z})$ iff $\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y})$. For the “moreover part”, by adding dummy variables, we can assume that \bar{z} corresponds to \bar{d} and $p(\bar{z}) \in S_{\bar{d}}(\mathfrak{C})$, where \bar{d} is a tuple of all elements of \mathfrak{C} in which each element is repeated infinitely many times. Finally, we can identify the variables in the tuple \bar{z} which correspond to the same elements in \bar{d} , and we obtain a shorter tuple, still denoted by \bar{z} , corresponding to \bar{c} and a type, still denoted by $p(\bar{z})$, in $S_{\bar{c}}(\mathfrak{C})$ which satisfy the requirements.

(ii) follows by adding dummy parameters, and (iii) by adding dummy variables. \square

The next remark explains that Definition 3.1 coincides with the usual definition of [externally] definable map from a type-definable set to a compact, Hausdorff space (in our case this space is finite). Recall that a function $f : X \rightarrow C$, where X is a type-definable subset of a sufficiently saturated model and C is a compact, Hausdorff space, is said to be [externally] definable if the preimages of any two disjoint closed subsets of C can be separated by a relatively [externally] definable subset of X . In particular, if C is finite, then this is equivalent to saying that all fibers of f are relatively [externally] definable subsets of X .

- Remark 3.4.* (i) For a coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$, c is [externally] definable in the sense of Definition 3.1 iff it is [externally] definable in the above sense (i.e. the fibers are relatively [externally] definable subsets of the type-definable set $\binom{\mathfrak{C}}{\bar{a}}$).
- (ii) If $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow r$ is [externally] definable in the above sense, then it remains so as a coloring with the target space 2^n , where $2^n \geq r$. So we can always assume that $r = 2^n$. \square

3.1. Ramsey properties. Motivated by the embedding Ramsey property for Fraïssé structures, we introduce the following natural notion.

Definition 3.5. A theory T has *EERP* (the *elementary embedding Ramsey property*) if for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$ and any $r < \omega$ there exists a finite subset $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}} \rightarrow r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\binom{C}{\bar{a}}$ is monochromatic with respect to c .

Remark 3.6. The definition of *EERP* does not depend on the choice of the monster (or just an \aleph_0 -saturated) model \mathfrak{C} of T , i.e. *EERP* is indeed a property of T . \square

Remark 3.7. Definition 3.5 generalizes the definition of the embedding Ramsey property for Fraïssé structures in the following sense: If K is an \aleph_0 -saturated Fraïssé structure, then K has the embedding Ramsey property iff $\text{Th}(K)$ has *EERP*.

Proof. Recall that if K is an \aleph_0 -saturated Fraïssé structure, then $\text{Th}(K)$ has q.e., so $\binom{C}{\bar{a}}^{\text{qf}} = \binom{C}{\bar{a}}$ for any finite $\bar{a}, C \subseteq K$. Therefore, we conclude using Fact 2.12 and Remark 3.6. \square

The next remark is standard and follows from [17, Remark 3.1].

Remark 3.8. A theory T has *EEERP* iff for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $r < \omega$, and any coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c . \square

In [17], the class of first order structures with *ERP* (*the embedding Ramsey property*) is introduced. A first order structure M has *ERP* if for any finite $\bar{a} \subseteq \bar{b} \subseteq M$ and any $r < \omega$ there exists a finite subset $C \subseteq M$ such that for any coloring $c : \binom{C}{\bar{a}}^{\text{Aut}} \rightarrow r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}^{\text{Aut}}$ such that $\binom{\bar{b}'}{\bar{a}}^{\text{Aut}}$ is monochromatic with respect to c . Here, for finite $\bar{a} \subseteq M$ and $C \subseteq M$:

$$\binom{C}{\bar{a}}^{\text{Aut}} := \{\bar{a}' \subseteq C \mid \bar{a}' = f(\bar{a}) \text{ for some } f \in \text{Aut}(M)\}.$$

For Fraïssé structures the next fact is one of the main results from [14], which was later generalized to arbitrary locally finite ultrahomogeneous structures in [26]. The formulation below comes from [17], but it can be checked (by passing to canonical ultrahomogeneous expansions and using an argument as in Fact 2.12) that it is equivalent to the one from [26].

Fact 3.9 ([17, Theorem 3.2]). *A first order structure M has ERP iff $\text{Aut}(M)$ is extremely amenable as a topological group.* \square

Note that if M is strongly \aleph_0 -homogeneous, then $\binom{C}{\bar{a}}^{\text{Aut}} = \binom{C}{\bar{a}}$, so if we in addition assume that M is \aleph_0 -saturated, then by Remark 3.6, M has *ERP* iff $\text{Th}(M)$ has *EEERP*. Hence, by Fact 3.9, we obtain the following.

Corollary 3.10. *For a theory T , $\text{Aut}(M)$ is extremely amenable (as a topological group) for some \aleph_0 -saturated and strongly \aleph_0 -homogeneous model $M \models T$ iff it is so for all \aleph_0 -saturated and strongly \aleph_0 -homogeneous models of T .* \square

We now introduce and study two new classes of theories by restricting our considerations to definable and externally definable colorings, which makes the whole subject more general and more model-theoretic.

Definition 3.11. A theory T has *[E]DEERP* (*the [externally] definable elementary embedding Ramsey property*) if for any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$ and any [externally] definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c .

Proposition 3.12. *The previous definitions do not depend on the choice of the monster (or just an \aleph_0 -saturated) model, i.e. DEERP and EDEERP are indeed properties of T .*

Proof. We fix the following notation. For any finite $\bar{a} \subseteq \bar{b}$ in a model of T , let \bar{x}' be some variables corresponding to \bar{b} and denote by $V_{\bar{a}, \bar{b}}$ the set of all $\bar{x} \subseteq \bar{x}'$ corresponding to the elementary copies of \bar{a} within \bar{b} . Note that $V_{\bar{a}, \bar{b}}$ depends only on $\text{tp}(\bar{a})$, $\text{tp}(\bar{b})$ and the choice of \bar{x}' .

In order to see the *DEERP* case, it is enough to note that (even assuming only that \mathfrak{C} is \aleph_0 -saturated) the statement defining *DEERP* inside \mathfrak{C} is equivalent to the following condition: for any finite $\bar{a} \subseteq \bar{b}$ (in any model of T), any formula

$\varphi(\bar{x}') \in \text{tp}(\bar{b})$, and any formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ without parameters with \bar{x} corresponding to \bar{a} :

$$T \vdash (\forall \bar{y})(\exists \bar{x}') \left(\varphi(\bar{x}') \wedge \bigwedge_{\bar{x}_1, \bar{x}_2 \in V_{\bar{a}, \bar{b}}} \bigwedge_{i < n} (\varphi_i(\bar{x}_1, \bar{y}) \leftrightarrow \varphi_i(\bar{x}_2, \bar{y})) \right).$$

We now turn to *EDEERP*. Suppose that \mathfrak{C} is a monster (or \aleph_0 -saturated) model which satisfies the property given in the externally definable case of Definition 3.11. It suffices to prove that any monster (or \aleph_0 -saturated) model \mathfrak{C}^* such that $\mathfrak{C}^* \prec \mathfrak{C}$ or $\mathfrak{C}^* \succ \mathfrak{C}$ satisfies it as well.

Let $\mathfrak{C}^* \prec \mathfrak{C}$. Consider any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}^*$, $n < \omega$, and an externally definable coloring $c^* : \binom{\mathfrak{C}^*}{\bar{a}} \rightarrow 2^n$ given by formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and a type $p^*(\bar{y}) \in S_{\bar{y}}(\mathfrak{C}^*)$ (see Remark 3.3). By Fact 2.25, let $p(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ be a strong heir extension of $p^*(\bar{y})$, and let $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ be the externally definable extension of c^* given by $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and $p(\bar{y})$. For a color $\varepsilon \in 2^n$ consider the formula $\theta_\varepsilon(\bar{x}, \bar{y}) := \bigwedge_{\bar{x} \in V_{\bar{a}, \bar{b}}} \bigwedge_{i < n} \varphi_i(\bar{x}, \bar{y})^{\varepsilon(i)}$. Note that for $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$, $c[\binom{\bar{b}'}{\bar{a}}] = \{\varepsilon\}$ iff $\theta_\varepsilon(\bar{b}', \bar{y}) \in p(\bar{y})$. By assumption, there is a color $\varepsilon \in 2^n$ and $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $c[\binom{\bar{b}'}{\bar{a}}] = \{\varepsilon\}$, hence $\theta_\varepsilon(\bar{b}', \bar{y}) \in p(\bar{y})$. Since $p^*(\bar{y}) \subseteq p(\bar{y})$ is a strong heir extension, there is $\bar{b}'' \subseteq \mathfrak{C}^*$ such that $\bar{b}'' \equiv \bar{b}'$ and $\theta_\varepsilon(\bar{b}'', \bar{y}) \in p^*(\bar{y})$. But this means that $\bar{b}'' \in \binom{\mathfrak{C}^*}{\bar{b}}$ and $c[\binom{\bar{b}''}{\bar{a}}] = \{\varepsilon\}$, so $\binom{\bar{b}''}{\bar{a}}$ is monochromatic with respect to c , and hence with respect to c^* . The case $\mathfrak{C}^* \succ \mathfrak{C}$ is very easy. \square

The following remark describes the connections between the introduced notions. Both implications below are strict: the lack of the converse of the first implication is witnessed by the theory of the random graph (see Example 5.6) or by $T := ACF_0$ with named constants from the algebraic closure of \mathbb{Q} (see example 5.2), and the second one by the theory of a certain random hypergraph (see Example 5.7).

Remark 3.13. For every theory T , $EERP \implies EDEERP \implies DEERP$.

We now turn to dynamical characterizations of theories with *DEERP* and *EDEERP*. Let T be a theory. In order to unify some proofs below, it is convenient to work with local versions of $[E]DEERP$. We say that a finitary type $q \in S(\emptyset)$ has $[E]DEERP$ if for some (equivalently, any) $\bar{a} \models q$, any finite $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , any $n < \omega$ and any [externally] definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c . Clearly, a theory T has $[E]DEERP$ iff every finitary type has it. For $q \in S_{\bar{x}}(T)$, a type $p \in S_{\bar{c}}(\mathfrak{C})$ is q -invariant if for every formula $\varphi(\bar{x}, \bar{y})$ and $\bar{a}, \bar{a}' \models q$: $\varphi(\bar{a}, \bar{y}) \in p$ iff $\varphi(\bar{a}', \bar{y}) \in p$ holds. The set of all q -invariant types from $S_{\bar{c}}(\mathfrak{C})$ is denoted by $\text{Inv}_{q, \bar{c}}(\mathfrak{C})$. Clearly, $\text{Inv}_{q, \bar{c}}(\mathfrak{C})$ is a subflow of $S_{\bar{c}}(\mathfrak{C})$, so for every $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$:

$$(\dagger) \quad \eta[\text{Inv}_{q, \bar{c}}(\mathfrak{C})] \subseteq \text{Inv}_{q, \bar{c}}(\mathfrak{C}).$$

Theorem 3.14. (i) A finitary type q has *EDEERP* iff there is $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta) \subseteq \text{Inv}_{q, \bar{c}}(\mathfrak{C})$.

(ii) T has *EDEERP* iff there exists $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$.

Proof. (i) (\implies) Suppose that q has *EDEERP* and take $\bar{a} \models q$. Fix a finite $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , $L(\bar{a})$ -formulae $\varphi_i(\bar{a}, \bar{y})$ and types $p_i \in S_{\bar{c}}(\mathfrak{C})$, $i < n$. Consider the externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given via $\varphi_i(\bar{x}, \bar{y})$'s and p_i 's. By

EDEERP there is $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ with $\binom{\bar{b}'}{\bar{a}}$ being monochromatic; take $\sigma \in \text{Aut}(\mathfrak{C})$ such that $\sigma(\bar{b}') = \bar{b}$. Then for $i < n$ and $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$ we have $\varphi_i(\bar{a}, \bar{y}) \in \sigma(p_i)$ iff $\varphi_i(\bar{a}', \bar{y}) \in \sigma(p_i)$.

Consider the family \mathcal{F} of pairs $(\bar{b}, \{(\varphi_i(\bar{y}), p_i)\}_{i < n})$, for finite $\bar{b} \supseteq \bar{a}$, $n < \omega$, $p_i \in S_{\bar{c}}(\mathfrak{C})$ and $\varphi_i(\bar{y}) \in L(\bar{a})$ for $i < n$. We order \mathcal{F} naturally by:

$$(\bar{b}, \{(\varphi_i(\bar{y}), p_i)\}_{i < n}) \leq (\bar{b}', \{(\varphi'_i(\bar{y}), p'_i)\}_{i < n'})$$

iff $\bar{b} \subseteq \bar{b}'$, $n \leq n'$ and $\{(\varphi_i(\bar{y}), p_i)\}_{i < n} \subseteq \{(\varphi'_i(\bar{y}), p'_i)\}_{i < n'}$; clearly, \mathcal{F} is directed by \leq . Consider a net $(\sigma_f)_{f \in \mathcal{F}}$ of automorphisms, where each σ_f is chosen by the previous paragraph for $f \in \mathcal{F}$. It is easy to check that any accumulation point $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ of this net satisfies our requirements.

(\Leftarrow) Let $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ be such that $\text{Im}(\eta) \subseteq \text{Inv}_{q, \bar{c}}(\mathfrak{C})$. Take $\bar{a} \models q$ and finite $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , $n < \omega$, and an externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given by formulae $\varphi_i(\bar{x}, \bar{y})$, $i < n$, and type $p \in S_{\bar{c}}(\mathfrak{C})$ (see Remark 3.3). Since $\eta(p)$ is q -invariant, $(\varphi_i(\bar{a}, \bar{y}) \leftrightarrow \varphi_i(\bar{a}', \bar{y})) \in \eta(p)$ holds for all $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ and $i < n$. This is an open condition on η , so there is $\sigma \in \text{Aut}(\mathfrak{C})$ such that $(\varphi_i(\bar{a}, \bar{y}) \leftrightarrow \varphi_i(\bar{a}', \bar{y})) \in \sigma(p)$ for all $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ and $i < n$. For $\bar{b}' := \sigma^{-1}(\bar{b})$ we get that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c .

(ii) (\Leftarrow) is clear by (i), so we prove (\Rightarrow). Suppose that T has *EDEERP*. By (i), for every finitary type q , choose $\eta_q \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(\eta_q) \subseteq \text{Inv}_{q, \bar{c}}(\mathfrak{C})$. For a finite set $S = \{q_1, \dots, q_n\}$ of finitary types from $S(\emptyset)$ put $\eta_S := \eta_{q_1} \circ \dots \circ \eta_{q_n}$ (the order of the composition is irrelevant). By the choice of η_{q_i} and (\dagger), we obtain $\text{Im}(\eta_S) \subseteq \bigcap_{i=1}^n \text{Inv}_{q_i, \bar{c}}(\mathfrak{C})$. Let S be the family of all finite sets of finitary types in $S(\emptyset)$. Direct S by inclusion, and let η be an accumulation point of the obtained net of the η_S 's. One easily concludes that $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$. \square

Corollary 3.15. *If T has *EDEERP*, then any minimal left ideal $\mathcal{M} \triangleleft \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ is trivial, hence $u\mathcal{M}$ (i.e. the Ellis group of T) and $\text{Gal}_L(T) = \text{Gal}_{\text{KP}}(T)$ are trivial as well.*

Proof. Assume that T has *EDEERP*. Let $\mathcal{M} \triangleleft \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ be a minimal left ideal and let $\eta_0 \in \mathcal{M}$ be arbitrary. By Theorem 3.14, there exists $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$. Set $\eta_1 = \eta\eta_0$; clearly $\eta_1 \in \mathcal{M}$ and $\text{Im}(\eta_1) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$. Then $\text{Aut}(\mathfrak{C})\eta_1 = \{\eta_1\}$, so $\{\eta_1\}$ is a minimal subflow. Therefore, $\mathcal{M} = \{\eta_1\}$, and so $u\mathcal{M}$ is trivial. Triviality of $\text{Gal}_L(T)$ follows from the existence of an epimorphism $u\mathcal{M} \rightarrow \text{Gal}_L(T)$ found in [18]. \square

Theorem 3.16. (i) *For a finitary type q , TFAE:*

- (1) q has *DEERP*.
- (2) There is an element $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\text{Inv}_{q, \bar{c}}(\mathfrak{C})$.
- (3) $\text{Inv}_{q, \bar{c}}(\mathfrak{C}) \neq \emptyset$

(ii) *For a theory T , TFAE:*

- (1) T has *DEERP*,
- (2) There is an element $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\text{Inv}_{\bar{c}}(\mathfrak{C})$,
- (3) T is extremely amenable, that is $\text{Inv}_{\bar{c}}(\mathfrak{C}) \neq \emptyset$.

Proof. (i) For (1) \Rightarrow (2) it suffices to note that by working with the realized types in $S_{\bar{c}}(\mathfrak{C})$ in place of arbitrary types $p_i \in S_{\bar{c}}(\mathfrak{C})$, the construction in the proof of

Theorem 3.14(i, \Rightarrow) yields η mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\text{Inv}_{q,\bar{c}}(\mathfrak{C})$. The implication (2) \Rightarrow (3) is obvious, so it remains to prove (3) \Rightarrow (1).

Assume that $\text{Inv}_{q,\bar{c}}(\mathfrak{C}) \neq \emptyset$ and take any $p \in \text{Inv}_{q,\bar{c}}(\mathfrak{C})$. Fix $\bar{a} \models q$. Consider any finite $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , $n < \omega$, and a definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given by $\varphi_i(\bar{x}, \bar{d})$, $i < n$, with \bar{d} finite. Let $p_0(\bar{y}) \in S_{\bar{y}}(\bar{b})$ be the restriction of p , where \bar{y} are the variables corresponding to \bar{d} within \bar{c} , and note that, by q -invariance of p , p_0 contains $\varphi_i(\bar{a}, \bar{y}) \leftrightarrow \varphi_i(\bar{a}_0, \bar{y})$ for all $\bar{a}_0 \in \binom{\bar{b}}{\bar{a}}$ and $i < n$. Take $\bar{d}_0 \models p_0$ and $f \in \text{Aut}(\mathfrak{C})$ such that $f(\bar{d}_0) = \bar{d}$; let $\bar{a}' = f(\bar{a})$ and $\bar{b}' = f(\bar{b})$. Then $\text{tp}(\bar{d}/\bar{b}')$ contains $\varphi_i(\bar{a}', \bar{y}) \leftrightarrow \varphi_i(\bar{a}_0, \bar{y})$ for all $\bar{a}_0 \in \binom{\bar{b}'}{\bar{a}'}$ and $i < n$. This means that $\binom{\bar{b}'}{\bar{a}'}$ is monochromatic with respect to c , as $\binom{\bar{b}'}{\bar{a}'} = \binom{\bar{b}}{\bar{a}}$.

(ii) (2) \Rightarrow (3) is obvious, and (3) \Rightarrow (1) follows by (i). To prove (1) \Rightarrow (2), for every finite $\bar{a} \subseteq \mathfrak{C}$ choose, by (i), $\eta_{\bar{a}} \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\text{Inv}_{\text{tp}(\bar{a}),\bar{c}}(\mathfrak{C})$. Direct all finite tuples \bar{a} from \mathfrak{C} by inclusion and take any accumulation point η of the net of the $\eta_{\bar{a}}$'s. It is easy to see that η maps all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\text{Inv}_{\bar{c}}(\mathfrak{C})$. \square

Proposition 3.12 together with the observation that in the proofs of Theorems 3.14 and 3.16 it is enough to assume only \aleph_0 -saturation and strong \aleph_0 -homogeneity of \mathfrak{C} and consider *EDEERP* [resp. *DEERP*] only in the chosen model \mathfrak{C} yield that both the existence of $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C})$ as well as extreme amenability of T are independent of the choice of the \aleph_0 -saturated and strongly \aleph_0 -homogeneous model \mathfrak{C} . On the other hand, the fact that extreme amenability of T is absolute was easily observed directly in [14], so, using the above observation on the proof of Theorem 3.16, we get the first part of Proposition 3.12, i.e. absoluteness of *DEERP* (at least for \aleph_0 -saturated and strongly \aleph_0 -homogeneous models).

Note that *DEERP* implies that $\text{Gal}_L(T)$ is trivial, because, by Theorem 3.16, T is extremely amenable and so $\text{Gal}_L(T)$ is trivial by [12, Proposition 4.2]. However, in contrast with *EDEERP*, Examples 5.7 and 5.8 show that a theory with *DEERP* need not have trivial or even finite Ellis group.

In Corollaries 2.17 and 2.19, we saw that in an $(m+1)$ -ary theory, in order to compute the Ellis group or to test the existence of an element in the Ellis semigroup with image contained in invariant types, we can restrict ourselves to the $\text{Aut}(\mathfrak{C})$ -flow $S_m(\mathfrak{C})$. The next proposition shows a similar behavior of *EDEERP*.

Proposition 3.17. *If T is $(m+1)$ -ary and each type from $S_m(T)$ has *EDEERP*, then T has *EDEERP*.*

Proof. The construction from Theorem 3.14(ii, \Rightarrow), but working with types $q_i \in S_m(T)$ instead of arbitrary finitary types, yields $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta) \subseteq \bigcap_{q \in S_m(T)} \text{Inv}_{q,\bar{c}}(\mathfrak{C})$. By $(m+1)$ -arity of T , this intersection equals $\text{Inv}_{\bar{c}}(\mathfrak{C})$, so T has *EDEERP* by Theorem 3.14(ii). \square

3.2. Separately finite externally definable elementary embedding Ramsey degree. By a direct analogy with the introduced notions of elementary embedding Ramsey properties, one may define the notions of finite elementary embedding Ramsey degrees. We introduce here theories with separately finite *EERdeg* and a weak form of separately finite *EDEERdeg*, as they play an essential role in this paper. In order to state this weak version, we need to use externally definable Δ -colorings (see Definition 3.1 and Remark 3.3).

- Definition 3.18.** (a) A theory T has *separately finite EERdeg* (*separately finite elementary embedding Ramsey degree*) if for any finite tuple \bar{a} there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$.
- (b) A theory T has *separately finite EDEERdeg* (*separately finite externally definable elementary embedding Ramsey degree*) if for any finite tuple \bar{a} , finite set of formulae Δ in variables \bar{x} (where $|\bar{x}| = |\bar{a}|$) and \bar{y} , and $q \in S_{\bar{y}}(\emptyset)$, there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing \bar{a} , $n < \omega$, and externally definable (Δ, q) -coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$. By Remark 3.3(iii), wlog we can assume here that $|\bar{y}| = |\bar{c}|$ and $q := \text{tp}(\bar{c}/\emptyset)$, so c is then an externally definable Δ -coloring.

The word “separately” is used here to stress that l depends on \bar{a} (in (b), l also depends on (Δ, q) , or only on Δ if $q = \text{tp}(\bar{c}/\emptyset)$). The least such number will be called the [*externally definable*] *elementary embedding Ramsey degree* of \bar{a} [with respect to (Δ, q) , or Δ if $q = \text{tp}(\bar{c}/\emptyset)$].

As in Remark 3.8, a theory T has sep. fin. *EERdeg* iff the following holds: For any finite tuple \bar{a} there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ and any $r < \omega$ there exists a finite subset $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}} \rightarrow r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$. By using this characterization, as in Remark 3.6 one shows that the property of having sep. fin. *EERdeg* does not depend on the choice of the monster (or just an \aleph_0 -saturated) model \mathfrak{C} , i.e. it is indeed a property of the theory. The counterpart of Remark 3.7 also holds:

Remark 3.19. Definition 3.18(a) generalizes the definition of sep. fin. embedding Ramsey degree for Fraïssé structures in the following sense: If K is an \aleph_0 -saturated Fraïssé structure, then it has sep. fin. embedding Ramsey degree iff $\text{Th}(K)$ has sep. fin. *EERdeg*. \square

The property of having sep. fin. *EDEERdeg* is absolute, too, i.e. does not depend on the choice of the monster model. To see this, one should follow the lines of the proof of absoluteness of *EDEERP* in Proposition 3.12 with the following modifications. Fix \bar{a} , Δ and q as in the definition of *EDEERdeg*. For any finite $\bar{b} \supseteq \bar{a}$ consider the same $V_{\bar{a}, \bar{b}}$ as in the proof of Proposition 3.12. For any $n < \omega$ and formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y}) \in \Delta$ consider the formula:

$$\theta(\bar{x}', \bar{y}_0, \dots, \bar{y}_{n-1}) := \bigvee_{\substack{V \subseteq V_{\bar{a}, \bar{b}} \\ |V|=l}} \bigwedge_{\bar{x}_0 \in V_{\bar{a}, \bar{b}}} \bigwedge_{\bar{x} \in V} \bigwedge_{i < n} \varphi_i(\bar{x}_0, \bar{y}_i) \leftrightarrow \varphi_i(\bar{x}, \bar{y}_i),$$

where l is the *EDEERdeg* of \bar{a} with respect to (Δ, q) computed in \mathfrak{C} , and the \bar{y}_i 's are pairwise disjoint copies of \bar{y} . Note that for an externally definable (Δ, q) -coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given by $\varphi_i(\bar{x}, \bar{y})$ and $p_i(\bar{y}) \in S_q(\mathfrak{C})$, $i < n$, for $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ we have: $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$ iff $\theta(\bar{b}', \bar{y}_0, \dots, \bar{y}_{n-1})$ is consistent with $\bigcup_{i < n} p_i(\bar{y}_i)$, iff $\bigcup_{i < n} p_i(\bar{y}_i) \vdash \theta(\bar{b}', \bar{y}_0, \dots, \bar{y}_{n-1})$. Using this, one can argue as in the proof of Proposition 3.12, starting from $p_i^*(\bar{y}) \in S_q(\mathfrak{C}^*)$, $i < n$, choosing a completion $p^*(\bar{y}_0, \dots, \bar{y}_{n-1}) \in S(\mathfrak{C}^*)$ of $\bigcup_{i < n} p_i^*(\bar{y}_i)$, and using a strong heir extension $p \in S(\mathfrak{C})$ of p^* .

Note that the diagram in Remark 3.13 expands to:

$$\begin{array}{ccccc} \text{EERP} & \Rightarrow & \text{EDEERP} & \Rightarrow & \text{DEERP} \\ \downarrow & & \downarrow & & \\ \text{sep. fin. EERdeg} & \Rightarrow & \text{sep. fin. EDEERdeg} & & \end{array}$$

All the implications written above are strict. The converse of the first vertical implication fails for the random graph; the converse of the second one fails for the hypergraph from Example 5.7; the converse of the lower horizontal implication fails e.g. in ACF_0 (see Example 5.2).

We now prove the counterpart of Theorem 3.14 for sep. finite $EDEERdeg$, which is one of the main results of this paper. For a formula $\varphi(\bar{x}, \bar{y})$ put $\varphi^{\text{opp}}(\bar{y}, \bar{x}) := \varphi(\bar{x}, \bar{y})$. For $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ put $\Delta^{\text{opp}} := \{\varphi_i^{\text{opp}}(\bar{y}, \bar{x})\}_{i < k}$.

Theorem 3.20. *T has sep. fin. $EDEERdeg$ iff for every $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ and $\bar{p} = \{p_j\}_{j < m} \subseteq S_{\bar{y}}(T)$ there exists $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ such that $\text{Im}(\eta)$ is finite.*

Proof. (\Rightarrow) Suppose that T has sep. fin. $EDEERdeg$. Fix any $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ and $\bar{p} = \{p_j\}_{j < m} \subseteq S_{\bar{y}}(T)$, where \bar{x} corresponds to \bar{c} and \bar{y} is finite. Fix $\bar{a}_j \models p_j$ for $j < m$, and put $\bar{a} = (\bar{a}_0, \dots, \bar{a}_{m-1})$. Let $\Delta' := \{\phi_{i,j}(\bar{z}, \bar{x}) \mid i < k, j < m\}$, where $\phi_{i,j}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and $\bar{z} = (\bar{y}_0, \dots, \bar{y}_{m-1})$ with \bar{y}_i corresponding to \bar{a}_i . Let $l < \omega$ be the $EDEERdeg$ of \bar{a} with respect to Δ' .

For any $n < \omega$, $q_0, \dots, q_{n-1} \in S_{\bar{c}}(\mathfrak{C})$, and finite $\bar{b} \supseteq \bar{a}$ consider the externally definable Δ' -coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^{kmn}$ given by:

$$c(\bar{a}')(i, j, t) = \begin{cases} 1, & \varphi_i(\bar{x}, \bar{a}'_j) \in q_t \\ 0, & \neg \varphi_i(\bar{x}, \bar{a}'_j) \in q_t \end{cases},$$

where $\bar{a}'_j \subseteq \bar{a}'$ is the subtuple corresponding to \bar{y}_j . (Note that c is indeed an externally definable Δ' -coloring via the formulae $\phi_{i,j,t}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and types $r_{i,j,t}(\bar{x}) := q_t(\bar{x})$ for $i < k$, $j < m$, and $t < n$.) By the choice of l , we can find $\bar{b}' \in \binom{\mathfrak{C}}{\bar{a}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$. Let $\sigma_{\bar{b}, \bar{q}} \in \text{Aut}(\mathfrak{C})$ be such that $\sigma_{\bar{b}, \bar{q}}(\bar{b}') = \bar{b}$; here \bar{q} denotes (q_0, \dots, q_{n-1}) . Consider the naturally directed family \mathcal{F} of all pairs (\bar{b}, \bar{q}) , and let η be an accumulation point of the net $(\sigma_{\bar{b}, \bar{q}})_{(\bar{b}, \bar{q}) \in \mathcal{F}}$ in $\text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$.

We claim that $\text{Im}(\eta)$ is finite. Suppose not, i.e. we can find $q_0, q_1, \dots \in S_{\bar{c}}(\mathfrak{C})$ such that the $\eta(\hat{q}_t)$'s are pairwise distinct, where $\hat{q}_t \in S_{\bar{c}, \Delta}(\bar{p})$ denotes the restriction of q_t . For each $t \neq t'$ the fact that $\eta(\hat{q}_t) \neq \eta(\hat{q}_{t'})$ is witnessed by one of the formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{k-1}(\bar{x}, \bar{y})$ and a realization of one of the types p_0, \dots, p_{m-1} . By Ramsey theorem, passing to a subsequence, we can assume that there are $i_0 < k$ and $j_0 < m$ such that for each $t \neq t'$ the fact that $\eta(\hat{q}_t) \neq \eta(\hat{q}_{t'})$ is witnessed by $\varphi_{i_0}(\bar{x}, \bar{y})$ and a realization of p_{j_0} . Let $n > 2^l$, and for $t < t' < n$ denote by $\bar{a}_{j_0}^{t, t'}$ a realization of p_{j_0} such that $\varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \eta(\hat{q}_t)$ iff $\neg \varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \eta(\hat{q}_{t'})$. Note that this is an open condition on η , so if we let $\bar{a}^{t, t'}$ to be a copy of \bar{a} extending $\bar{a}_{j_0}^{t, t'}$, then we can find a tuple (\bar{b}, \bar{q}) greater than $(\bar{a} \frown (\bar{a}^{t, t'})_{t < t' < n}, (q_0, \dots, q_{n-1}))$ in \mathcal{F} such that $\varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \sigma_{\bar{b}, \bar{q}}(\hat{q}_t)$ iff $\neg \varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \sigma_{\bar{b}, \bar{q}}(\hat{q}_{t'})$. By the choice of $\sigma_{\bar{b}, \bar{q}}$, $\bar{b}' = \sigma_{\bar{b}, \bar{q}}^{-1}(\bar{b})$ satisfies $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$.

For $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$ let $S(\bar{a}') := \{t < n \mid \varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_t\}$. Since $\#c[\binom{\bar{b}'}{\bar{a}}] \leq l$, we have $\#\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} \leq l$. Put $l' = \#\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\}$, so $l' \leq l$, and let $\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} = \{S_0, \dots, S_{l'-1}\}$. Note that the mapping $f : n \rightarrow \mathcal{P}(\{S_0, \dots, S_{l'-1}\})$ given by $f(t) = \{S_u \mid u < l', t \in S_u\}$ is injective. Indeed, for $t < t' < n$, by the choice of \bar{b} , we have $\bar{a}^{t, t'} \subseteq \bar{b}$, so $\bar{a}' := \sigma_{\bar{b}, \bar{q}}^{-1}(\bar{a}^{t, t'}) \subseteq \bar{b}'$. But since $\varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \sigma_{\bar{b}, \bar{q}}(\hat{q}_t)$ iff $\neg \varphi_{i_0}(\bar{x}, \bar{a}_{j_0}^{t, t'}) \in \sigma_{\bar{b}, \bar{q}}(\hat{q}_{t'})$, we have $\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in \hat{q}_t$ iff $\neg \varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in \hat{q}_{t'}$, i.e. $\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_t$ iff $\neg \varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_{t'}$, hence $t \in S(\bar{a}')$ iff $t' \notin S(\bar{a}')$. Thus, $S(\bar{a}')$

distinguishes $f(t)$ and $f(t')$, and f is injective. Therefore, $n \leq 2^{t'} \leq 2^l$, which contradicts the choice of n .

(\Leftarrow) Suppose now that the right hand side of the theorem holds. Take a finite tuple $\bar{a} \subseteq \mathfrak{C}$ and a finite set of formulae Δ in variables \bar{x}, \bar{y} , where \bar{y} corresponds to \bar{c} ; let $\#\Delta = k$ and set $p = \text{tp}(\bar{a})$. By assumption, we may find $\eta \in \text{EL}(S_{\bar{c}, \Delta^{\text{opp}}}(p))$ with $\text{Im}(\eta)$ finite, say $\#\text{Im}(\eta) = t$. We claim that $l := 2^{kt}$ satisfies our requirements for \bar{a} and Δ .

Fix a finite \bar{b} containing \bar{a} , $n < \omega$, and an externally definable Δ -coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given by $\varphi_i(\bar{x}, \bar{y}) \in \Delta$ and $q_i(\bar{y}) \in S_{\bar{c}}(\mathfrak{C})$, $i < n$. Actually, we may assume that the types $q_i(\bar{y})$'s are from $S_{\bar{c}, \Delta^{\text{opp}}}(\mathfrak{C})$ by taking their appropriate restrictions, as the coloring c depends only on these restrictions. For each $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ find $\varepsilon_{\bar{a}'} \in 2^n$ such that $\varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'(i)}} \in \eta(q_i(\bar{y}))$ for all $i < n$. Note that we have only 2^{kt} possibilities for $\varepsilon_{\bar{a}'}$, as we only have k -many formulae in Δ and t -many types in $\text{Im}(\eta)$ (namely, if $\varphi_i(\bar{x}, \bar{y}) = \varphi_{i'}(\bar{x}, \bar{y})$ and $\eta(q_i(\bar{y})) = \eta(q_{i'}(\bar{y}))$, then, by consistency, we must have $\varepsilon_{\bar{a}'}(i) = \varepsilon_{\bar{a}'}(i')$). Consider the open condition on η given by: $\varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'(i)}} \in \eta(q_i(\bar{y}))$ for all $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ and $i < n$. Let $\sigma \in \text{Aut}(\mathfrak{C})$ be such that: $\varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'(i)}} \in \sigma(q_i(\bar{y}))$, i.e. $\varphi_i(\sigma^{-1}(\bar{a}'), \bar{y})^{\varepsilon_{\bar{a}'(i)}} \in q_i(\bar{y})$, for all $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ and $i < n$. Put $\bar{b}' = \sigma^{-1}(\bar{b})$; we have: $\varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'(i)}} \in q_i(\bar{y})$ for all $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$ and $i < n$. But since we only had 2^{kt} possibilities for ε 's, this means that $\#c[\binom{\bar{b}'}{\bar{a}}] \leq 2^{kt}$. \square

In fact, a slight elaboration on our proof yields concrete bounds.

Corollary 3.21. (i) Let $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$ and $\bar{p} = \{p_j\}_{j < m} \subseteq S_{\bar{y}}(T)$, where $|\bar{x}| = |\bar{c}|$ and \bar{y} is finite. Fix $\bar{a}_j \models p_j$ for $j < m$, and put $\bar{a} = (\bar{a}_0, \dots, \bar{a}_{m-1})$. Let $\Delta' := \{\phi_{i,j}(\bar{z}, \bar{x}) \mid i < k, j < m\}$, where $\phi_{i,j}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and $\bar{z} = (\bar{y}_0, \dots, \bar{y}_{m-1})$. Assume $l < \omega$ is the *EDEERdeg* of \bar{a} with respect to Δ' . Then there exists $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ such that $\#\text{Im}(\eta) \leq 2^{mkl}$.

(ii) Let \bar{a} be a finite tuple and $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i < k}$, where $|\bar{y}| = |\bar{c}|$. Put $p = \text{tp}(\bar{a})$. Assume that there is $\eta \in \text{EL}(S_{\bar{c}, \Delta^{\text{opp}}}(p))$ with $\#\text{Im}(\eta) = t < \omega$. Then the *EDEERdeg* of \bar{a} with respect to Δ is bounded by 2^{kt} .

Proof. (ii) was obtained in the proof of (\Leftarrow) in Theorem 3.20. To get (i), one needs to elaborate on the proof of (\Rightarrow) as follows. We do not use Ramsey theorem. Instead, suppose we have n types $q_0, \dots, q_{n-1} \in S_{\bar{c}}(\mathfrak{C})$ with the $\eta(\hat{q}_t)$'s pairwise distinct. Then for any $t < t' < n$ we can find $i(t, t') < k$, $j(t, t') < m$, and a realization $\bar{a}_{j(t, t')}^{t, t'}$ of $p_{j(t, t')}$ such that $\varphi_{i(t, t')}(\bar{x}, \bar{a}_{j(t, t')}^{t, t'}) \in \eta(\hat{q}_t)$ iff $\neg \varphi_{i(t, t')}(\bar{x}, \bar{a}_{j(t, t')}^{t, t'}) \in \eta(\hat{q}_{t'})$. Then we continue with obvious adjustments until the definition of $S(\bar{a}')$. Next, for any $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$, $i < k$, and $j < m$ put $S(\bar{a}')^{i, j} := \{t < n \mid \varphi_i(\bar{x}, \bar{a}'_j) \in q_t\}$; define $S(\bar{a}') := (S(\bar{a}')^{i, j})_{i < k, j < m}$. Then still $\#\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} \leq l$, so $\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} = \{S_0, \dots, S_{l'-1}\}$ for some $l' \leq l$. Let $f : n \rightarrow (2^{k \times m})^{l'}$ be defined by saying that $f(t)$ is the unique function $\delta : l' \rightarrow 2^{k \times m}$ such that for all $u < l'$, $i < k$, $j < m$ one has $\delta(u)(i, j) = 1 \iff t \in S_u^{i, j}$, where $S_u^{i, j}$ is the (i, j) -th coordinate of S_u . As before, one shows that f is injective, which clearly implies that $n \leq 2^{kml}$. \square

The fact that the property of having sep. fin. *EDEERdeg* does not depend on the choice of \mathfrak{C} together with the fact that in the proof of Theorem 3.20 we work in the given \mathfrak{C} imply that the right hand side of Theorem 3.20 does not depend on the choice of \mathfrak{C} , too. By examining the above proofs, one can also see that it is enough to assume here only that \mathfrak{C} is \aleph_0 -saturated and strongly \aleph_0 -homogeneous.

The main consequence of Theorem 3.20 will be Corollary 4.1 which leads to many examples of theories with profinite Ellis group.

3.3. Elementary embedding convex Ramsey property. We briefly explain here the counterpart of Corollary 3.10 for amenability in place of extreme amenability, leaving some details to the reader.

In [20], Moore characterized amenability of the automorphism group of a Fraïssé structure via the so-called convex Ramsey property. This was generalized to arbitrary structures in [17], where the notion of *ECRP* (the *embedding convex Ramsey property*) for a first order structure is introduced. A structure M has *ECRP* if for any $\epsilon > 0$, any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq M$, and any $n < \omega$ there exists a finite subset $C \subseteq M$ such that for any coloring $c : \binom{C}{\bar{a}}^{\text{Aut}} \rightarrow 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \text{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$ and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}^{\text{Aut}}$ we have:

$$\max_{i < n} \left| \sum_{j < k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j < k} \lambda_j c(\sigma_j(\bar{a}''))(i) \right| \leq \epsilon.$$

Recall that for finite $\bar{a} \subseteq M$ and $C \subseteq M$, $\binom{C}{\bar{a}}^{\text{Aut}}$ denotes the set of all $f(\bar{a})$ for $f \in \text{Aut}(M)$ with $f(\bar{a}) \subseteq C$. Also, recall that if M is strongly \aleph_0 -homogeneous, then $\binom{C}{\bar{a}}^{\text{Aut}} = \binom{C}{\bar{a}}$. By [17, Remark 4.2], in the above definition we can equivalently remove the part “there is a finite subset $C \subseteq M$ such that” and everywhere replace C by M . The following fact was proved in [17].

Fact 3.22. [17, Theorem 4.3] *Let M be a first-order structure. TFAE:*

- (1) $\text{Aut}(M)$ is amenable as a topological group;
- (2) M has *ECRP*;
- (3) M has strong *ECRP*. □

By analogy with *EERP*, we introduce the following definition.

Definition 3.23. A theory T has *EECRP* (the *elementary embedding convex Ramsey property*) if it has a monster model \mathfrak{C} with *ECRP*.

A proof of the next lemma is left as an exercise.

Lemma 3.24. *If M and M^* are two elementarily equivalent \aleph_0 -saturated and strongly \aleph_0 -homogeneous models, then M has *ECRP* iff M^* does. Thus, a theory T has *EECRP* iff some (equiv. every) \aleph_0 -saturated and strongly \aleph_0 -homogeneous model of T has *ECRP*. □*

By Fact 3.22 and Lemma 3.24, we get the main conclusion of Subsection 3.3.

Corollary 3.25. *For a theory T , $\text{Aut}(M)$ is amenable (as a topological group) for some \aleph_0 -saturated and strongly \aleph_0 -homogeneous model $M \models T$ iff it is amenable for all \aleph_0 -saturated and strongly \aleph_0 -homogeneous models of T . □*

3.4. Definable elementary embedding convex Ramsey property. By Theorem 3.16, a theory T is extremely amenable iff it has *DEERP*. In this subsection, we will get an analogous result for amenable theories (see Definition 2.26). The corresponding Ramsey property-like condition which describes amenability of a theory is the *definable elementary embedding convex Ramsey property*.

Definition 3.26. A theory T has *DEECRP* (the *definable elementary embedding convex Ramsey property*) iff for any $\epsilon > 0$, any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$, and any definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \text{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i < n} \left| \sum_{j < k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j < k} \lambda_j c(\sigma_j(\bar{a}''))(i) \right| \leq \epsilon.$$

A theory T has the *strong DEECRP* if the previous holds for $\epsilon = 0$.

Remark 3.27. By a direct argument, one may see that the previous definition does not depend on the choice of the monster model \mathfrak{C} of T . Moreover, it is enough to ask that \mathfrak{C} is \aleph_0 -saturated and strongly \aleph_0 -homogeneous.

Theorem 3.28. *For a theory T the following conditions are equivalent:*

- (1) T is amenable;
- (2) T has *DEECRP*;
- (3) T has *strong DEECRP*.

Proof. (3) \Rightarrow (2) is obvious.

(2) \Rightarrow (1) Assume that T has *DEECRP*. Toward a contradiction, suppose that for a finite \bar{d} there is no an invariant, finitely additive probability measure on the Boolean algebra of clopens in $S_{\bar{d}}(\mathfrak{C})$. By compactness, for some $\epsilon > 0$, formulae $\varphi_i(\bar{x}_i, \bar{y})$ and tuples $\bar{a}_i \equiv \bar{a}'_i$, $i < n$, there is no measure $\mu \in \mathfrak{M}_{\bar{d}} := \mathfrak{M}_{\text{tp}(\bar{d})}$ (see Subsection 2.6) satisfying:

$$(\dagger) \quad |\mu([\varphi_i(\bar{a}_i, \bar{y})]) - \mu([\varphi_i(\bar{a}'_i, \bar{y})])| \leq \epsilon \text{ for all } i < n.$$

By using dummy variables we may assume that all \bar{x}_i 's are mutually equal, as well as all \bar{a}_i 's; denote them by \bar{x} and \bar{a} , respectively.

Set $\bar{b} = \bar{a} \hat{\ } \bar{a}'_0 \hat{\ } \dots \hat{\ } \bar{a}'_{n-1}$. Consider the definable coloring c on $\binom{\mathfrak{C}}{\bar{a}}$ given by the $\varphi_i(\bar{x}, \bar{d})$'s. By *DEECRP*, there are $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\sum \lambda_i = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \text{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ and for any $i < n$ we have $|\sum_{j < k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j < k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leq \epsilon$.

Consider $\mu = \lambda_0 \text{tp}(\sigma_0^{-1}(\bar{d})/\mathfrak{C}) + \dots + \lambda_{k-1} \text{tp}(\sigma_{k-1}^{-1}(\bar{d})/\mathfrak{C})$, defined by:

$$\mu([\phi(\bar{y})]) := \sum_{j < k} \{\lambda_j \mid \phi(\bar{y}) \in \text{tp}(\sigma_j^{-1}(\bar{d})/\mathfrak{C})\},$$

for $\phi(\bar{y}) \in L(\mathfrak{C})$; it is easy to see that $\mu \in \mathfrak{M}_{\bar{d}}$, as well as that for $i < n$, $j < k$, and $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ we have $c(\sigma_j(\bar{a}'))(i) = 1$ iff $\varphi_i(\bar{a}', \bar{y}) \in \text{tp}(\sigma_j^{-1}(\bar{d})/\mathfrak{C})$, and thus, $\sum_{j < k} \lambda_j c(\sigma_j(\bar{a}'))(i) = \mu([\varphi_i(\bar{a}', \bar{y})])$. Therefore, μ satisfies (\dagger) . A contradiction.

(1) \Rightarrow (3) Assume that T is amenable. Fix $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and a definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \rightarrow 2^n$ given by $\varphi_0(\bar{x}, \bar{d}), \dots, \varphi_{n-1}(\bar{x}, \bar{d})$. By amenability, we can find an invariant, finitely additive probability measure μ on the clopens of $S_{\bar{d}}(\mathfrak{C})$. Let $\binom{\bar{b}}{\bar{a}} = \{\bar{a}_s\}_{s < t}$ and let us consider clopens $[\varphi_i(\bar{a}_s, \bar{y})]$ in $S_{\bar{d}}(\mathfrak{C})$ for $i < n$, $s < t$. Let \mathcal{F} be the family of all $\varepsilon \in 2^{n \times t}$ such that $\psi_\varepsilon(\bar{y}) := \bigwedge_{s < t} \varphi_i(\bar{a}_s, \bar{y})^{\varepsilon(i,s)}$ is consistent with $\text{tp}(\bar{d})$. (Note that the $[\psi_\varepsilon(\bar{y})]$'s are the atoms of the Boolean algebra generated by the $[\varphi_i(\bar{a}_s, \bar{y})]$'s in $S_{\bar{d}}(\mathfrak{C})$.) Set $k = \#\mathcal{F}$ and $\lambda_\varepsilon = \mu([\psi_\varepsilon(\bar{y})])$ for $\varepsilon \in \mathcal{F}$; clearly $\sum_{\varepsilon \in \mathcal{F}} \lambda_\varepsilon = 1$. Further, for each $\varepsilon \in \mathcal{F}$ choose $\sigma_\varepsilon \in \text{Aut}(\mathfrak{C})$ such that $\sigma_\varepsilon^{-1}(\bar{d}) \models \psi_\varepsilon(\bar{y})$. We claim that the chosen λ_ε 's and σ_ε 's satisfy our requirements.

First, note the following: for $s < t$ and $\varepsilon \in \mathcal{F}$, $c(\sigma_\varepsilon(\bar{a}_s))(i) = 1$ iff $\models \varphi_i(\sigma_\varepsilon(\bar{a}_s), \bar{d})$ iff $\models \varphi_i(\bar{a}_s, \sigma_\varepsilon^{-1}(\bar{d}))$ iff $\varepsilon(i, s) = 1$. Hence,

$$\sum_{\varepsilon \in \mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_s))(i) = \sum_{\substack{\varepsilon \in \mathcal{F} \\ \varepsilon(i, s) = 1}} \mu([\psi_\varepsilon(\bar{y})]) = \mu\left(\bigsqcup_{\substack{\varepsilon \in \mathcal{F} \\ \varepsilon(i, s) = 1}} [\psi_\varepsilon(\bar{y})]\right) = \mu([\varphi_i(\bar{a}_s, \bar{y})]),$$

so for any $s < s' < t$ we have:

$$\sum_{\varepsilon \in \mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_s))(i) - \sum_{\varepsilon \in \mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_{s'}))(i) = \mu([\varphi_i(\bar{a}_s, \bar{y})]) - \mu([\varphi_i(\bar{a}_{s'}, \bar{y})]) = 0$$

by the invariance of μ . Thus, (3) holds. \square

As usual, Remark 3.27 together with the fact that in the proof of Theorem 3.28 we work in the given \mathfrak{C} imply that amenability of T is absolute (i.e. does not depend on the choice of \mathfrak{C}). But this was proved directly in [12], which together with the observation that in the proof of Theorem 3.28 we work in the given \mathfrak{C} implies Remark 3.27. By examining the above proofs, one can also see that we can assume here only that \mathfrak{C} is \aleph_0 -saturated and strongly \aleph_0 -homogeneous.

4. AROUND PROFINITENESS OF THE ELLIS GROUP

In this section, we will prove Theorem 3 as well as several other criteria for [pro]finiteness of the Ellis group of the theory in question.

Proof of Theorem 3. The implications (D) \implies (C) \implies (B) \implies (A) \implies (A') \implies (A'') follow from various facts or observations made so far: the first implication follows by Fact 2.1, the second one is clear by Lemma 2.21, the third one by Fact 2.8, and the last two by Proposition 2.13.

It remains to show (B) \implies (C). Denote by \mathcal{F} the set of all pairs (Δ, \bar{p}) , where Δ is a finite set of formulae and \bar{p} a finite set of types from $S(T)$ with the same variables as the parameter variables of the formulae in Δ . For any $t = (\Delta, \bar{p}) \in \mathcal{F}$ put $S_t := S_{\bar{c}, \Delta}(\bar{p})$.

Let $F : S_{\bar{c}}(\mathfrak{C}) \rightarrow \varprojlim_{t \in \mathcal{F}} S_t$ be the flow isomorphism given by restrictions, and let $G : S_{\bar{c}}(\mathfrak{C}) \rightarrow \varprojlim_{i \in I} X_i$ be a flow isomorphism. Also, let $f_{t_0} : \varprojlim_{t \in \mathcal{F}} S_t \rightarrow S_{t_0}$ and $g_{i_0} : \varprojlim_{i \in I} X_i \rightarrow X_{i_0}$ be projections. Consider: $F_t := \{(p, q) \in S_{\bar{c}}(\mathfrak{C})^2 \mid f_t \circ F(p) = f_t \circ F(q)\}$ and $G_i := \{(p, q) \in S_{\bar{c}}(\mathfrak{C})^2 \mid g_i \circ G(p) = g_i \circ G(q)\}$ for $t \in \mathcal{F}$ and $i \in I$. Both the F_t 's and the G_i 's are obviously equivalence relations on $S_{\bar{c}}(\mathfrak{C})$. They are closed: F_t is the inverse image of the diagonal on S_t under $f_t \circ F$, and similarly for the G_i 's. Moreover, they are clearly $\text{Aut}(\mathfrak{C})$ -invariant and also directed in the sense that $t \leq t' \in \mathcal{F}$ implies $F_{t'} \subseteq F_t$, and $i \leq i' \in I$ implies $G_{i'} \subseteq G_i$.

Notice that $\bigcap_{t \in \mathcal{F}} F_t = D$ and $\bigcap_{i \in I} G_i = D$, where $D = \{(p, p) \mid p \in S_{\bar{c}}(\mathfrak{C})\}$.

Claim. *For any $t \in \mathcal{F}$ there is $i \in I$ such that $G_i \subseteq F_t$.*

Proof of Claim. Observe that for any clopen $C \subseteq S_{\bar{c}}(\mathfrak{C})$ we can find $i_0 \in I$ such that C is a union of G_{i_0} -classes. To see this, note that $\bigcap_{i \in I} G_i = D$ implies that $\bigcup_{i \in I} G_i^c \cup (C \times C) \cup (C^c \times C^c) = S_{\bar{c}}(\mathfrak{C})^2$, where each member is open. By compactness, $\bigcup_{i \in I_0} G_i^c \cup (C \times C) \cup (C^c \times C^c) = S_{\bar{c}}(\mathfrak{C})^2$ for some finite I_0 , so $\bigcap_{i \in I_0} G_i \subseteq (C \times C) \cup (C^c \times C^c)$. By choosing i_0 to be greater than all elements of I_0 , we get $G_{i_0} \subseteq (C \times C) \cup (C^c \times C^c)$, and the conclusion follows.

Let $t = (\Delta, \bar{p})$, where $\Delta = \{\varphi_l(\bar{x}, \bar{y})\}_{l < k}$ and $\bar{p} = \{p_j(\bar{y})\}_{j < m} \subseteq S_{\bar{y}}(T)$. For $j < m$ choose $\bar{a}_j \models p_j$. Then $[\varphi_l(\bar{x}, \bar{a}_j)]$ is a clopen subset of S_t for $l < k$ and

$j < m$, so $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a}_j)]]$ is clopen in $S_{\bar{c}}(\mathfrak{C})$. By the previous paragraph, we can find $i_{l,j} \in I$ such that $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a}_j)]]$ is a union of $G_{i_{l,j}}$ -classes. Since $G_{i_{l,j}}$ is $\text{Aut}(\mathfrak{C})$ -invariant and $f_t \circ F$ is a homomorphism of flows, we get that for every $\bar{a} \models p_j$, $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a})]]$ is a union of $G_{i_{l,j}}$ -classes. Let $i \in I$ be greater than or equal to all $i_{l,j}$'s, for $l < k$ and $j < m$. It follows that for every clopen C in S_t we have that $(f_t \circ F)^{-1}[C]$ is a union of G_i -classes.

Therefore, for any $q \in S_t$, since $(f_t \circ F)^{-1}[\{q\}] = \bigcap_C (f_t \circ F)^{-1}[C]$, with the intersection taken over all clopens C containing q , we get that $(f_t \circ F)^{-1}[\{q\}]$ is a union of G_i -classes. This means that any F_t -class is a union of G_i -classes, hence $G_i \subseteq F_t$. \square Claim

By the claim, for any $t \in \mathcal{F}$ we can find $i \in I$ such that the identity map $S_{\bar{c}}(\mathfrak{C}) \rightarrow S_{\bar{c}}(\mathfrak{C})$ induces a flow epimorphism $S_{\bar{c}}(\mathfrak{C})/G_i \rightarrow S_{\bar{c}}(\mathfrak{C})/F_t$. On the other hand, $S_{\bar{c}}(\mathfrak{C})/G_i \cong X_i$ and $S_{\bar{c}}(\mathfrak{C})/F_t \cong S_t$ as flows. Therefore, there exists a flow epimorphism $X_i \rightarrow S_t$, and we are done by Fact 2.5(i) and Fact 2.3. \square

By Theorem 3.20, for a theory with sep. finite *EDEERdeg* Condition (D) holds. Hence, since (D) \implies (A), we obtain:

Corollary 4.1. *If T has sep. fin. *EDEERdeg*, then the Ellis group of T is profinite.* \square

As mentioned in the introduction, we will give examples showing that (A'') does not imply (A') (see Example 5.11) and (A') does not imply (B) (see Example 5.10). We do not know however whether Example 5.10 satisfies (A), so we do not know whether it shows that (A') does not imply (A) or that (A) does not imply (B). We have also not found an example showing that (C) does not imply (D).

We do not expect that (A) \implies (C) is true. However, we can easily see that (A) is equivalent to a weaker version of (C):

(C') for every finite set of formulae Δ and types $\bar{p} \subseteq S(T)$, the Ellis group of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}, \Delta}(\bar{p}))$ is profinite.

By Lemma 2.21 and Fact 2.8, we see that (C') \implies (B') \implies (A), where (B') is the weaker version of (B) in which we require only profiniteness of the Ellis groups of the flows X_i .

Proposition 4.2. *Conditions (A), (B'), and (C') are equivalent.*

Proof. It remains to prove (A) \implies (C'). Consider an $\text{Aut}(\mathfrak{C})$ -flow epimorphism $S_{\bar{c}}(\mathfrak{C}) \rightarrow S_{\bar{c}, \Delta}(\bar{p})$ given by the restriction. By Fact 2.5, it induces an $\text{Aut}(\mathfrak{C})$ -flow and semigroup epimorphism $f : \text{EL}(S_{\bar{c}}(\mathfrak{C})) \rightarrow \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$. Let \mathcal{M} be a minimal left ideal of $\text{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $u \in \mathcal{J}(\mathcal{M})$. By Fact 2.3, $\mathcal{M}' := f[\mathcal{M}]$ is a minimal left ideal of $\text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$, $u' := f(u) \in \mathcal{J}(\mathcal{M}')$, and $f_{\upharpoonright u\mathcal{M}} : u\mathcal{M} \rightarrow u'\mathcal{M}'$ is a group epimorphism and quotient map (in the τ -topology). By Remark 2.2(b), it is enough to prove that $u'\mathcal{M}'$ is 0-dimensional. For this, it suffices to prove that $f_{\upharpoonright u\mathcal{M}}[U]$ is clopen for any clopen $U \subseteq u\mathcal{M}$ (as $u\mathcal{M}$ is 0-dimensional), which boils down to showing that $f_{\upharpoonright u\mathcal{M}}^{-1}[f_{\upharpoonright u\mathcal{M}}[U]]$ is clopen. Since $f_{\upharpoonright u\mathcal{M}}^{-1}[f_{\upharpoonright u\mathcal{M}}[U]] = \ker(f_{\upharpoonright u\mathcal{M}}) \cdot U$, this set is clopen by the fact that $u'\mathcal{M}'$ is T_1 , compactness of $u\mathcal{M}$, and continuity of multiplication in $u\mathcal{M}$. \square

We now state two criteria which in some quite common situations guarantee profiniteness (or even finiteness) of the Ellis group.

Proposition 4.3. *Suppose that $L \subseteq L^*$ are finite relational languages, T^* is a complete L^* -theory with q.e. such that $T := T^*_{\uparrow L}$ also has q.e. Let \mathfrak{C}^* be a monster model of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster model of T . If there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$, then for any finite \bar{z} there exists $\eta \in \text{EL}(S_{\bar{z}}(\mathfrak{C}))$ such that $\text{Im}(\eta)$ is finite. In particular, the Ellis group of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{z}}(\mathfrak{C}))$ is finite.*

We should stress that here we consider two different flows: $(\text{Aut}(\mathfrak{C}^*), S_{\bar{c}}(\mathfrak{C}^*))$ and $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, defined with respect to two different languages. In order to avoid confusions, we denote by asterisk notions defined with respect to the language L^* , and without asterisk notions defined with respect to the language L .

Proof. By Lemma 2.20(iv), there is $\eta \in \text{EL}(S_{\bar{z}}(\mathfrak{C}))$ with $\text{Im}(\eta) = \text{Inv}_{\bar{z}}^*(\mathfrak{C})$, where $\text{Inv}_{\bar{z}}^*(\mathfrak{C}) \subseteq S_{\bar{z}}(\mathfrak{C})$ is the set of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types. Hence, since T has q.e., each type $p \in \text{Im}(\eta)$ is determined by saying whether $R(\bar{z}, \bar{b}) \in p$ or not for some (every) $\bar{b} \models q$, for all $R(\bar{z}, \bar{y}) \in L$ and $q \in S_{\bar{y}}(T^*)$. On the other hand, $S_{\bar{y}}(T^*)$ is finite by q.e. of T^* and finiteness of L^* . Since L is also finite, we see that we have only finitely many possibilities for $p \in \text{Im}(\eta)$, and thus $\text{Im}(\eta)$ is finite. (Implicitly we also used here that the languages are relational.)

The ‘‘In particular’’ part follows by Fact 2.1. \square

Corollary 4.4. *Under the assumptions of Proposition 4.3, the Ellis group of the flow $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ is finite.*

Proof. Since L is a finite relational language and T has q.e., T is m -ary, where m is the maximal arity of the symbols in L . By Corollary 2.17, the Ellis groups of the flows $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ and $(\text{Aut}(\mathfrak{C}), S_{m-1}(\mathfrak{C}))$ are isomorphic. The latter is finite by Proposition 4.3. \square

Proposition 4.5. *Suppose that $L \subseteq L^*$, T^* is a complete L^* -theory, and $T := T^*_{\uparrow L}$. Let \mathfrak{C}^* be a monster of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster of T . Assume that for any finitely many variables \bar{y} there are only finitely many extensions in $S_{\bar{y}}(T^*)$ of each type from $S_{\bar{y}}(T)$. If there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$, then (D) holds. In particular, the Ellis group of T is profinite.*

Proof. The existence of $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ with $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$ implies, by Lemma 2.20(ii), that there is $\eta \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(\eta) \subseteq \text{Inv}_{\bar{c}}^*(\mathfrak{C})$, where $\text{Inv}_{\bar{c}}^*(\mathfrak{C}) \subseteq S_{\bar{c}}(\mathfrak{C})$ is the set of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types. Then, for each finite Δ, \bar{p} , the induced $\eta_{\Delta, \bar{p}} \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ satisfies $\text{Im}(\eta_{\Delta, \bar{p}}) \subseteq \text{Inv}_{\bar{c}, \Delta}^*(\bar{p})$, where $\text{Inv}_{\bar{c}, \Delta}^*(\bar{p})$ is the set of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types from $S_{\bar{c}, \Delta}(\bar{p})$.

Consider any finite Δ and \bar{p} . For each $q_{\Delta, \bar{p}} \in \text{Inv}_{\bar{c}, \Delta}^*(\bar{p})$ and $\varphi(\bar{x}, \bar{y}) \in \Delta$ we have: $\varphi(\bar{x}, \bar{b}) \in q_{\Delta, \bar{p}}(\bar{x})$, where $\text{tp}(\bar{b}) \in \bar{p}$, iff $\varphi(\bar{x}, \bar{b}') \in q_{\Delta, \bar{p}}(\bar{x})$ for all \bar{b}' realizing the same extension in $S_{\bar{y}}(T^*)$ of $\text{tp}(\bar{b}) \in S_{\bar{y}}(T)$ as \bar{b} . Since Δ and \bar{p} are finite and each $p_0 \in \bar{p}$ has only finitely many extensions in $S_{\bar{y}}(T^*)$, $q_{\Delta, \bar{p}}(\bar{x})$ is completely determined by finitely many conditions. Therefore, $\text{Inv}_{\bar{c}, \Delta}^*(\bar{p})$ is finite, and so $\text{Im}(\eta_{\Delta, \bar{p}})$ is finite.

The ‘‘In particular’’ part follows from (D), as (D) \implies (A). \square

Remark 4.6. The assumption of Proposition 4.5 that for each finite \bar{y} there are only finitely many extensions in $S_{\bar{y}}(T^*)$ of each type from $S_{\bar{y}}(T)$ holds when L^* is relational, $L^* \setminus L$ is finite, and T^* has q.e. \square

Corollary 4.7. *The assumptions of Proposition 4.3 imply the assumptions of Proposition 4.5 which in turn imply that T has sep. fin. EDEERdeg.*

Proof. The first part follows from Remark 4.6. To see the rest, note that since Proposition 4.5 yields (D), we can use Theorem 3.20(\Leftarrow). \square

Thus, the assumptions of Proposition 4.5 yield a criterion for having sep. fin. *EDEERdeg*. In fact, this is a counterpart of the criterion proved by Zucker in [31, Theorem 8.14], which we now explain (but for the definitions the reader is referred to [31] and [23]). Zucker shows that a Fraïssé structure has sep. fin. Ramsey degree iff it has a Fraïssé, precompact (relational) expansion whose age has the embedding Ramsey property and the expansion property relative to the age of the original structure. Combining this with [23, Theorem 6] (and noting that the Fraïssé subclass of $\text{Age}(\mathbf{F}^*)$ obtained there is a reasonable expansion of $\text{Age}(\mathbf{F})$), we get that a Fraïssé structure has sep. fin. Ramsey degree iff it has a Fraïssé, precompact (relational) expansion whose age has the embedding Ramsey property. Now, if both the original and the expanded structure are \aleph_0 -saturated (so have q.e.), then being precompact exactly means that for any finitely many variables \bar{y} there are only finitely many extensions in $S_{\bar{y}}(T^*)$ of each type from $S_{\bar{y}}(T)$. Moreover, by Theorem 3.14, the assumption of Proposition 4.5 saying that there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathcal{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathcal{C}^*)$ is equivalent to T^* having *EDEERP* (which is the appropriate counterpart of the embedding Ramsey property).

However, Example 5.3 shows that, in contrast with Zucker's result, in our case the assumptions of Proposition 4.5 are only sufficient to have sep. finite *EDEERdeg* for T , but they are not necessary. It could be interesting to find an “iff” criterion of this kind; we have not tried to do that.

5. APPLICATIONS AND EXAMPLES

In this section, we use our analysis to prove [pro]finiteness (or triviality) of the Ellis groups in some classes of theories, as well as give examples illustrating various phenomena and showing the lack of implications between some key conditions considered in this paper.

Example 5.1. We prove that stable theories have sep. fin. *EDEERdeg*. Denote by $\text{NF}_{\bar{c}}(\mathcal{C})$ the space of all types from $S_{\bar{c}}(\mathcal{C})$ which do not fork over \emptyset . Using only one consequence of stability, namely that forking equals dividing over \emptyset , the proof of [15, Proposition 7.11] yields:

Claim. *There exists an idempotent u in a minimal left ideal \mathcal{M} of $\text{EL}(S_{\bar{c}}(\mathcal{C}))$ with $\text{Im}(u) \subseteq \text{NF}_{\bar{c}}(\mathcal{C})$.*

Choose $u \in \mathcal{J}(\mathcal{M})$ given by the claim. Fix a finite set of L -formulae Δ and finite tuple of types \bar{p} . Let $u_{\Delta, \bar{p}} := \hat{f}(u) \in \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$, where $\hat{f} : \text{EL}(S_{\bar{c}}(\mathcal{C})) \rightarrow \text{EL}(S_{\bar{c}, \Delta}(\bar{p}))$ is induced by the restriction map $f : S_{\bar{c}}(\mathcal{C}) \rightarrow S_{\bar{c}, \Delta}(\bar{p})$ via Fact 2.5(i). Denote by $\text{NF}_{\bar{c}, \Delta}(\bar{p})$ the restrictions of all types from $\text{NF}_{\bar{c}}(\mathcal{C})$ to $S_{\bar{c}, \Delta}(\bar{p})$. By Corollary 2.6, we conclude that $\text{Im}(u_{\Delta, \bar{p}}) \subseteq \text{NF}_{\bar{c}, \Delta}(\bar{p})$. On the other hand, by finiteness of the Δ -multiplicity of $\text{tp}(\bar{c})$, $\text{NF}_{\bar{c}, \Delta}(\bar{p})$ is finite. Hence, $\text{Im}(u_{\Delta, \bar{p}})$ is finite. Thus, Condition (D) is fulfilled, and hence T has sep. fin. *EDEERdeg* by Theorem 3.20.

If we in addition assume that $\text{tp}(\bar{c})$ is stationary (equivalently, all types in $S(\emptyset)$ are stationary), then T has *EDEERP*. Indeed, under this assumption, $\text{NF}_{\bar{c}}(\mathcal{C})$ is a singleton consisting of an invariant type, so u given by the claim has image contained in the invariant types, and hence T has *EDEERP* by Theorem 3.14.

Conversely, if $\text{tp}(\bar{c})$ is not stationary, then there is no global invariant extension of this type, i.e. T is not extremely amenable, so T does not have *DEERP*.

This implies that in stable theories the properties *EDEERP* and *DEERP* are equivalent (which is also immediate from definitions, using definability of types).

The next concrete example shows that a stable theory does not need to have sep. fin. *ERdeg*; in particular, sep. fin. *EDEERdeg* does not imply sep. fin. *ERdeg*.

Example 5.2. Take $T := \text{ACF}_0$. Consider on \mathfrak{C}^\times the relations \sim_n given by $a \sim_n a'$ iff $a' = \xi_n^k a$ for some k , where ξ_n denotes an n -th primitive root of unity. The \sim_n 's are equivalence relations on \mathfrak{C}^\times , and let X_n be a set of representatives of the \sim_n -classes; then $\mathfrak{C}^\times = \bigsqcup_{k < n} \xi_n^k X_n$. For any transcendental a , take $\bar{b} = (a, \xi_n a, \dots, \xi_n^{n-1} a)$ and consider the coloring $c : \binom{\mathfrak{C}}{a} \rightarrow n$ given by: $c(a') = k$ iff $a' \in \xi_n^k X_n$. For any $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ we have that $\bar{b}' = (a', \omega_n a', \dots, \omega_n^{n-1} a')$ for some transcendental a' and primitive n -th root of unity ω_n . Note that $\#c[\binom{\bar{b}'}{a}] = n$. Thus, a transcendental element cannot have finite *ERdeg*, so T does not have sep. fin. *ERdeg*, while T is stable and as such has sep. fin. *EDEERdeg*.

If we modify T by naming all constants from the algebraic closure of \mathbb{Q} , then all types in $S(\emptyset)$ are stationary, so the resulting theory has *EDEERP*, but the above argument shows that it does not have sep. fin. *ERdeg*.

The next example shows that the criterion for having sep. fin. *EDEERdeg* given in Proposition 4.5 (see also Corollary 4.7) is not a necessary condition.

Example 5.3. Let T be a stable theory with a finitary type $p \in S_{\bar{y}}(T)$ of infinite multiplicity (i.e. with infinitely many global non-forking extensions). (For example, one can take $T := (\mathbb{Z}, +)$ and $p(y) := \text{tp}(1)$.) Then it has sep. fin. *EDEERdeg*, but we will show that there is no expansion T^* of T satisfying the assumptions of Proposition 4.5. Suppose for a contradiction that T^* is such an expansion. The assumption that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$ implies that T^* is extremely amenable, so $\text{acl}^{eq,*}(\emptyset) = \text{dcl}^{eq,*}(\emptyset)$ (both computed in \mathfrak{C}^*). Since p has infinite multiplicity, it has infinitely many extensions to complete types over $\text{acl}^{eq}(\emptyset)$ computed in \mathfrak{C} , so also over $\text{acl}^{eq,*}(\emptyset) \supseteq \text{acl}^{eq}(\emptyset)$. Therefore, we conclude that p has infinitely many extensions in $S_{\bar{y}}(T^*)$, a contradiction.

In the following example, we list some Fraïssé classes which are known to have the embedding Ramsey property and whose Fraïssé limits are \aleph_0 -categorical, and hence \aleph_0 -saturated. By Remark 3.7, the theories of these Fraïssé limits have *ERRP*, so *EDEERP* as well, and thus their Ellis groups are trivial by Corollary 3.15.

By a hypergraph we mean a structure in a finite relational language such that each basic relation R is irreflexive (i.e. $(R(a_0, \dots, a_{n-1}))$ implies that the a_i 's are pairwise distinct) and symmetric (i.e. invariant under permutations of coordinates).

- Example 5.4.** (a) The class of all finite (linearly) ordered hypergraphs omitting a fixed class of finite irreducible hypergraphs in a finite relational language containing \leq . See [21, Theorem A]. In particular, the theories of the ordered random graph and the ordered K_n -free random graph have *ERRP*.
- (b) The class of all finite sets with n linear orderings (for a fixed n) in the language $L = \{\leq_1, \dots, \leq_n\}$. See [28, Theorem 4].
- (c) The class of all finite structures in the language $L = \{\sqsubseteq, \leq\}$ for which \sqsubseteq is a partial ordering and \leq is a linear ordering extending \sqsubseteq . See [27].

- (d) The class of all finite structures in the language $L = \{\sqsubseteq, \leq, \prec\}$ for which \sqsubseteq is a partial ordering, \leq is a linear ordering, and \prec is a linear ordering extending \sqsubseteq . See [28, Theorem 1].
- (e) The class of all naturally ordered finite vector spaces over a fixed finite field F in the language $L = \{+, m_a, \leq\}_{a \in F}$, where $+$ is the addition, m_a is the unary multiplication by $a \in F$, and \leq is the anti-lexicographical linear ordering induced by an ordering of a basis. See [14] together with [9, Corollary 2].
- (f) The class of all naturally ordered finite Boolean algebras in the language of Boolean algebras expanded by \leq , where \leq is the anti-lexicographical linear order induced by an ordering of atoms. See [14, Proposition 6.13] and [10].
- (g) The class of all finite linearly ordered structures in the infinite language consisting of relational symbols R_n , $n > 0$, where R_n is n -ary, such that each R_n is irreflexive and symmetric. It is easy to see that this is a Fraïssé class whose limit is \aleph_0 -categorical. The fact that it has the embedding Ramsey property follows from the fact that the restrictions to the finite sublanguages have it by (a). This holds more generally in a situation when for each n there are only finitely many relational symbols of arity n .

In the examples of Fraïssé structures with the embedding Ramsey property which are not \aleph_0 -saturated, the situation is not so obvious, as we cannot use Remark 3.7 to deduce *EERP* and so *EDEERP*. We leave as a problem to find an example of a Fraïssé structure with the embedding Ramsey property whose theory does not have *EERP* or even *EDEERP*, or show that there is none.

We now list some Fraïssé classes which are known to have sep. fin. embedding Ramsey degree and whose Fraïssé limits are \aleph_0 -categorical, and hence \aleph_0 -saturated. By Remark 3.19, the theories of these Fraïssé limits have sep. fin. *EERdeg*, so sep. fin. *EDEERdeg*, hence their Ellis groups are profinite by Corollary 4.1.

As was recalled after Corollary 4.7, by [31] and [23], to see that a Fraïssé class \mathcal{K} has sep. fin. embedding Ramsey degree, one needs to find a Fraïssé class which is a reasonable, precompact, relational expansion of \mathcal{K} with the embedding Ramsey property.

- Example 5.5.** (a) The class of all finite hypergraphs omitting a fixed class of finite irreducible hypergraphs in a finite relational language. Example 5.4(a) gives the desired expansion of this class.
- (b) The class of all finite vector spaces over a fixed finite field. Example 5.4(e) gives the desired expansion of this class.
 - (c) The class of all finite Boolean algebras in the language of all Boolean algebras. Example 5.4(f) gives the desired expansion of this class.
 - (d) The age of any homogeneous directed graph. See [13].
 - (e) The class of all finite structures in the infinite language consisting of relational symbols R_n , $n > 0$, where R_n is n -ary, such that each R_n is irreflexive and symmetric. Example 5.4(g) gives the desired expansion of this class. This holds more generally in a situation when for each n there are only finitely many relational symbols of arity n .

Sometimes we can say more. For example, in Example 5.5(b) the Fraïssé limit is just an infinite dimensional vector space over a finite field, so it is strongly minimal with the property that $\text{NF}_{\bar{e}}(\mathcal{C}) = \text{Inv}_{\bar{e}}(\mathcal{C})$, and hence the Ellis group is trivial by the claim in Example 5.1 and the argument from the proof of Corollary 3.15. The

situation in Example 5.5(a) is more interesting. Namely, the Ellis group there is finite. This follows by Corollary 4.4. Indeed, let L be a finite relational language and $L^* = L \cup \{\leq\}$. Let K^* be the Fraïssé limit of the class of all linearly ordered finite L -hypergraphs (possibly omitting a fixed class of finite irreducible hypergraphs). Then $K := K^*_{\uparrow L}$ is the Fraïssé limit of all finite L -hypergraphs (omitting this fixed class of finite irreducible hypergraphs). We may find a monster model \mathfrak{C}^* of $T^* := \text{Th}(K^*)$ such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster model of $T := \text{Th}(K) = T^*_{\uparrow L}$. By Example 5.4(a), T^* has *EDEERP*, so we can find $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$ by Theorem 3.14. Moreover, both theories T and T^* have q.e. Thus, by Corollary 4.4, the Ellis group of T is finite. The same holds for the homogeneous directed graphs from Example 5.5(d).

Example 5.6. If T is the theory of the random n -hypergraph (so L in Example 5.5(a) consists of only one n -ary relational symbol R), then one can say even more: T has *EDEERP*, so the Ellis group is trivial. To see this, consider $L^* := L \cup \{\leq\}$. Let T be the theory of the random n -hypergraph (the theory of the Fraïssé limit of the class of all finite L -hypergraphs), and T^* the theory of the ordered random n -hypergraph (the theory of the Fraïssé limit of the class of all ordered finite n -hypergraphs); then $T = T^*_{\uparrow L}$. Choose a monster model $\mathfrak{C}^* \models T^*$ such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster model of T .

By Example 5.4(a), T^* has *EDEERP*, so, by Theorem 3.14, we can find $\eta_0^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta_0^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$. By Lemma 2.20(ii), there is $\eta_0 \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\text{Im}(\eta_0) \subseteq \text{Inv}_{\bar{c}}^*(\mathfrak{C})$, where $\text{Inv}_{\bar{c}}^*(\mathfrak{C})$ is the set of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types in $S_{\bar{c}}(\mathfrak{C})$. We finish by Theorem 3.14 after noting that each $\text{Aut}(\mathfrak{C}^*)$ -invariant type from $S_{\bar{c}}(\mathfrak{C})$ is also $\text{Aut}(\mathfrak{C})$ -invariant. To see this observe that, as T^* has q.e. and R is n -ary and irreflexive, for any \bar{a} and \bar{b} with $|\bar{a}| = |\bar{b}| < n$, $\bar{a} \equiv^{L^*} \bar{b}'$ for some permutation \bar{b}' of \bar{b} . So, by symmetry of R , for any $\text{Aut}(\mathfrak{C}^*)$ -invariant type $p(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$, $\bar{z} \subseteq \bar{x}$, and $\bar{a} \subseteq \mathfrak{C}$ with $|\bar{z}| + |\bar{a}| = n$ and $|\bar{a}| < n$, we have $R(\bar{z}, \bar{a}) \in p(\bar{x})$ iff $R(\bar{z}, \bar{b}) \in p(\bar{x})$. So, $p(\bar{x})$ is $\text{Aut}(\mathfrak{C})$ -invariant by q.e. of T .

A similar argument to the one in the previous example is viable if L contains finitely many relational symbols each of which is of one of the two consecutive arities n and $n + 1$. We leave to the reader to check the details. However, the above conclusion does not hold for all random hypergraphs in finite languages. In the following example, we state that the theory of the random $(2, 4)$ -hypergraph has a non-trivial Ellis group and describe various consequences of this result. All computations around this example are contained in Appendix A.

Example 5.7. Let $L = \{R_2, R_4\}$, where R_2 is a binary and R_4 is a quaternary relation. The Fraïssé limit K of the class of all finite L -hypergraphs is the random $(2, 4)$ -hypergraph. We will prove that the Ellis group of the theory $T := \text{Th}(K)$ is $\mathbb{Z}/2\mathbb{Z}$. Hence, T does not have *EDEERP* by Corollary 3.15, although it does have sep. fin. *EERdeg* by Example 5.5(a). Furthermore, this theory does have *DEERP* by Theorem 3.16, as the class above has free amalgamation and so T is extremely amenable (see the discussion after Corollary 2.16 in [12]). So the theory T shows that in general sep. fin. *EDEERdeg* (or even *EERdeg*) does not imply *EDEERP*, and that *DEERP* does not imply *EDEERP*.

Since T is extremely amenable, $\text{Gal}_L(T) = \text{Gal}_{KP}(T)$ is trivial by [12, Proposition 4.2], whereas the Ellis group of T is non-trivial. On the other hand, [15, Theorem 0.7] says that a certain natural epimorphism from the Ellis group of an

amenable theory with NIP to its KP-Galois group is always an isomorphism (this is a variant of Newelski's conjecture for groups of automorphisms). Thus, our theory T shows that in this theorem the assumption that the theory has NIP cannot be dropped (even assuming extreme amenability of the theory in question). Such an example was not known so far.

Now, we will describe a variant of the above example which yields a Fraïssé structure whose theory T satisfies the assumptions of Remark 4.6 and Proposition 4.5, and so has sep. fin. $EDEERdeg$, and the Ellis group of T is infinite (and clearly profinite). Also, T is extremely amenable, so $\text{Gal}_{KP}(T)$ is trivial. It will be a many-sorted example. But this is not a problem, because, as we said before, the whole theory developed in this paper works in a many-sorted context after minor adjustments. In the case of Proposition 4.5, it is easy to see that the proof goes through (only in Corollary 2.16, which is involved in this proof, one has to consider the tuple \bar{z} in which there are infinitely many variables associated with each sort).

Example 5.8. Consider the language L consisting of infinitely many sorts S_n , $n < \omega$, and relational symbols R_2^n and R_4^n , $n < \omega$, where each R_2^n is binary, each R_4^n quaternary, and they are both associated with the sort S_n . Let $L^* := L \cup \{\leq_n : n < \omega\}$, where each \leq_n is a binary symbol associated with S_n . Let K be the L^* -structure which is the disjoint union of copies of the ordered random $(2, 4)$ -hypergraph; then $K := K^*_{\uparrow L}$ is the disjoint union of copies of the random $(2, 4)$ -hypergraph. Put $T := \text{Th}(K)$ and $T^* := \text{Th}(K^*)$. Choose a monster model \mathfrak{C}^* of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\uparrow L}$ is a monster model of T .

Note that the assumptions of the obvious many-sorted version of Remark 4.6 are satisfied, so we check the remaining assumption of Proposition 4.5 that there is $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(\eta^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$. Let $\mathfrak{C}_n^* := S_n(\mathfrak{C}^*)$ be the $\{R_2^n, R_4^n, \leq_n\}$ -structure induced from \mathfrak{C}^* , for $n < \omega$. Then $\text{Aut}(\mathfrak{C}^*) = \prod_{n < \omega} \text{Aut}(\mathfrak{C}_n^*)$ and $S_{\bar{c}}(\mathfrak{C}^*) = \prod_{n < \omega} S_{\bar{c}_n}(\mathfrak{C}_n^*)$ after the clear identifications, where $\bar{c}_n \subseteq \bar{c}$ is an enumeration of \mathfrak{C}_n^* . By [25, Lemma 6.44], $\text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ is naturally isomorphic (as a semi-group and as an $\text{Aut}(\mathfrak{C}^*)$ -flow) with $\prod_{n < \omega} \text{EL}(S_{\bar{c}_n}(\mathfrak{C}_n^*))$. But in each $\text{EL}(S_{\bar{c}_n}(\mathfrak{C}_n^*))$ we have η_n^* whose image is contained in the $\text{Aut}(\mathfrak{C}_n^*)$ -invariant types (by Example 5.4(a) and Theorem 3.14), so the corresponding $\eta^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ has image contained in $\text{Inv}_{\bar{c}}(\mathfrak{C}^*)$.

Now, we will compute the Ellis group of T . Let $\mathfrak{C}_n := S_n(\mathfrak{C})$ be the $\{R_2^n, R_4^n\}$ -structure induced from \mathfrak{C} , for $n < \omega$. Then $\text{Aut}(\mathfrak{C}) = \prod_{n < \omega} \text{Aut}(\mathfrak{C}_n)$ and $S_{\bar{c}}(\mathfrak{C}) = \prod_{n < \omega} S_{\bar{c}_n}(\mathfrak{C}_n)$ after the clear identifications (where $\bar{c}_n \subseteq \bar{c}$ is an enumeration of \mathfrak{C}_n). By [25, Lemma 6.44], $u\mathcal{M} \cong \prod_{n < \omega} u_n\mathcal{M}_n$ (also in the τ -topologies), where \mathcal{M} and \mathcal{M}_n are minimal left ideals of $\text{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $\text{EL}(S_{\bar{c}_n}(\mathfrak{C}_n))$, respectively, and $u \in \mathcal{J}(\mathcal{M})$, $u_n \in \mathcal{J}(\mathcal{M}_n)$. Since each $u_n\mathcal{M}_n \cong \mathbb{Z}/2\mathbb{Z}$ by Example 5.7, we conclude that $u\mathcal{M} \cong (\mathbb{Z}/2\mathbb{Z})^\omega$ as a topological group; in particular, $u\mathcal{M}$ is infinite.

Extreme amenability of T can be seen directly: each finitary type $p(\bar{x}) \in S(\emptyset)$ has a global invariant extension determined by the formulae: $y \neq a$, $\neg R_2^n(y, b)$, and $\neg R_4^n(\bar{y}', \bar{b}')$, for all $n < \omega$, and $a, b, \bar{b}' \subseteq \mathfrak{C}$ (with $|\bar{b}'| \geq 1$), $y \in \bar{x}$, $\bar{y}' \subseteq \bar{x}$ (with $|\bar{y}'| \geq 1$) from the sorts for which these formulae make sense.

We now give an example of a theory with sep. fin. $EDEERdeg$ (even $EDEERP$) such that for some finite set of formulae Δ there is no $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\mathfrak{C}))$ with finite image. This shows that we really need to consider spaces $S_{\bar{c}, \Delta}(\bar{p})$, rather than the classical spaces of Δ -types $S_{\bar{c}, \Delta}(\mathfrak{C})$.

Example 5.9. Let $L = \{R_2, P_n\}_{n < \omega}$, where the P_n 's are unary and R_2 is binary. Put $P_\omega := \bigwedge_{n < \omega} \neg P_n$. Consider the class of all finite L -structures A such that:

- P_n^A are mutually disjoint for $n < \omega$, and
- R_2^A is irreflexive and symmetric.

This class is Fraïssé. Its Fraïssé limit K consists of infinitely many disjoint parts $P_n(K)$, for $n \leq \omega$, each of which is isomorphic to the random graph, but also there is the random interaction between them. Also, $T := \text{Th}(K)$ has q.e., hence it is binary, K is \aleph_0 -saturated, although it is not \aleph_0 -categorical.

To see that T has *EDEERP*, by Proposition 3.17, we shall prove that for each singleton a , finite \bar{b} containing a , $n < \omega$, and a coloring $c : \binom{\mathcal{C}}{a} \rightarrow 2^n$ there is $\bar{b}' \in \binom{\mathcal{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{a}$ is monochromatic with respect to c . For each $n \leq \omega$, $P_n(\mathcal{C})$ is the set of realizations of a complete 1-type over \emptyset . Fix $a \in \mathcal{C}$; then $a \in P_n(\mathcal{C})$ for some $n \leq \omega$. Take a finite $\bar{b} \ni a$ and without loss assume $\bar{b} \subseteq P_n(\mathcal{C})$. Take $r < \omega$ and $c : \binom{\mathcal{C}}{a} \rightarrow r$ (so $c : P_n(\mathcal{C}) \rightarrow r$). By saturation, there is a copy $G \subseteq P_n(\mathcal{C})$ of the random graph. The restriction of c to G corresponds to a finite partition of G , so there is a monochromatic isomorphic copy G' of G (see [3, Proposition 3.3]). We finish by taking a copy $\bar{b}' \subseteq G'$ of \bar{b} , as by q.e., $\bar{b}' \in \binom{\mathcal{C}}{\bar{b}}$.

Consider now $\Delta := \{R\}$. We check that $\text{EL}(S_{\bar{c}, \Delta}(\mathcal{C}))$ does not contain an element with finite image. Let \bar{x} correspond to \bar{c} and let $x_0 \in \bar{x}$ be any fixed single variable. For every $S \subseteq \omega + 1$ denote by $\pi_S(x_0)$ the set:

$$\{R(x_0, a) \mid a \in P_n(\mathcal{C}) \text{ for some } n \in S\} \cup \{\neg R(x_0, a) \mid a \in P_n(\mathcal{C}) \text{ for some } n \notin S\}.$$

By randomness, $\pi_S(x_0)$ is consistent with $\text{tp}(\bar{c})$, so we can find $p_S(\bar{x}) \in S_{\bar{c}, \Delta}(\mathcal{C})$ extending it. Each $\pi_S(x_0)$ is $\text{Aut}(\mathcal{C})$ -invariant, as the $P_n(\mathcal{C})$'s are sets of realizations of complete 1-types over \emptyset . Thus, for each $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\mathcal{C}))$ we have $\pi_S(x_0) \subseteq \eta(p_S)$, so $\eta(p_S)$ are pairwise distinct for $S \subseteq \omega + 1$. Therefore, $\text{Im}(\eta)$ is infinite.

Next, we give an example of a theory T showing that (A') does not imply (B) in Theorem 3. Hence, (D) fails for T , so T does not have sep. fin. *EDEERdeg* by Theorem 3.20. Also, T is amenable and so has *DEECRP* by Theorem 3.28.

Example 5.10. Consider the relational language $L = \{R, E_n\}_{n < \omega}$, where each symbol binary, and consider an L -structure M such that:

- R is irreflexive and symmetric;
- each E_n is an equivalence relation with exactly two classes;
- equivalences $\{E_n\}_{n < \omega}$ are independent in the sense that for each choice of E_n -class C_n , the family $\{C_n\}_{n < \omega}$ has the finite intersection property;
- for pairwise distinct $a_i, b_j \in M$, where $i < l, j < m$, and any $n < \omega$, the set $\bigcap_{i < l} R(M, a_i) \cap \bigcap_{j < m} \neg R(M, b_j)$ intersects each $\bigcap_{k \leq n} E_k$ -class.

Note that in particular we have that (M, R) is a model of the theory of the random graph. Let $T := \text{Th}(M)$. By standard arguments, T has q.e. Hence, there is only one type in $S_1(T)$. Let $\mathcal{C} \models T$ be a monster model.

We will first show that Condition (C) (equivalently (B)) fails for T . More precisely, we will show that the Ellis group of the $\text{Aut}(\mathcal{C})$ -flow $S_{\bar{c}, \Delta}(p)$ is infinite, where $\Delta := \{R(x_0, y)\}$ and $p(y) \in S_y(T)$ is the unique 1-type over \emptyset (so, $S_{\bar{c}, \Delta}(p) = S_{\bar{c}, \Delta}(\mathcal{C})$). Here, \bar{x} is reserved for \bar{c} and $x_0 \in \bar{x}$ is any fixed variable. Consider the following global partial types:

$$\pi_n(\bar{x}) := \{(R(x_0, a) \leftrightarrow R(x_0, b)) \leftrightarrow E_n(a, b) \mid a, b \in \mathcal{C}\}.$$

By randomness, each $\pi_n(\bar{x}) \cup \text{tp}(\bar{c})$ is consistent. So $X_n := [\pi_n(\bar{x})]$ is an $\text{Aut}(\mathfrak{C})$ -subflow of $S_{\bar{c}}(\mathfrak{C})$. By independence of the relations E_n , the X_n 's are pairwise disjoint. Let $\Phi : S_{\bar{c}}(\mathfrak{C}) \rightarrow S_{\bar{c}, \Delta}(\mathfrak{C})$ be the $\text{Aut}(\mathfrak{C})$ -flow epimorphism given by restriction to Δ -types. Note that $\Phi^{-1}[\Phi[X_n]] = X_n$. Therefore, $Y_n := \Phi[X_n]$ are pairwise disjoint $\text{Aut}(\mathfrak{C})$ -subflows of $S_{\bar{c}, \Delta}(\mathfrak{C})$. Hence, for any $\eta \in \text{EL}(S_{\bar{c}, \Delta}(\mathfrak{C}))$, $\eta[Y_n] \subseteq Y_n$, and so $\text{Im}(\eta)$ is infinite. Thus, Condition (D) does not hold, but we want to show that (C) fails.

Let $u \in \text{EL}(S_{\bar{c}, \Delta}(\mathfrak{C}))$ be an idempotent in a minimal left ideal \mathcal{M} , and let $q_n(\bar{x}) \in Y_n$ be in $\text{Im}(u)$; so $u(q_n) = q_n$. For each $n < \omega$ there is $\sigma_n \in \text{Aut}(\mathfrak{C})$ such that σ_n fixes each E_k -class for $k < n$ and swaps the E_n -classes. Fix $a \in \mathfrak{C}$ and $k < n$. For $\sigma \in \text{Aut}(\mathfrak{C})$, note that $R(x_0, \sigma_n^{-1}(\sigma^{-1}(a))) \in q_k$ iff $R(x_0, \sigma^{-1}(a)) \in q_k$ iff $R(x_0, \sigma_k^{-1}(\sigma^{-1}(a))) \notin q_k$. Thus $R(x_0, a) \in \sigma(\sigma_n(q_k))$ iff $R(x_0, a) \notin \sigma(\sigma_k(q_k))$ for every $\sigma \in \text{Aut}(\mathfrak{C})$, so $R(x_0, a) \in u(\sigma_n(q_k))$ iff $R(x_0, a) \notin u(\sigma_k(q_k))$ holds as well. This means that $u\sigma_n u(q_k) = u\sigma_n(q_k) \neq u\sigma_k(q_k) = u\sigma_k u(q_k)$, so $u\sigma_n u \neq u\sigma_k u$. Since all $u\sigma_n u \in u\mathcal{M}$, $u\mathcal{M}$ is infinite and Condition (C) does not hold.

We now prove amenability of T . We have to find an invariant, Borel probability measure on $S_p(\mathfrak{C})$, for every $p(\bar{y}) \in S_{\bar{y}}(T)$ where $\bar{y} = (y_0, \dots, y_{m-1})$. As the reduct T' of T to the sublanguage $L' := \{E_n\}_{n < \omega}$ is stable, it is amenable by [12, Corollary 2.21]. We can choose \mathfrak{C} so that $\mathfrak{C}' := \mathfrak{C}_{\upharpoonright L'}$ is a monster model of T' . Let $p' \in S_{\bar{y}}(T')$ be the restriction of p to the language L' . By amenability of T' , there is an $\text{Aut}(\mathfrak{C}')$ -invariant, Borel probability measure ν on $S_{p'}(\mathfrak{C}')$. Consider $\Phi : S_p(\mathfrak{C}) \rightarrow S_{p'}(\mathfrak{C}')$ given by restriction. Let $q_R(\bar{y}) := \{R(y_i, a) \mid i < m, a \in \mathfrak{C}\}$; by randomness and q.e., $q_R(\bar{y})$ is a partial type consistent with $p(\bar{y})$. It is $\text{Aut}(\mathfrak{C})$ -invariant, so $X := [q_R(\bar{y})]$ is an $\text{Aut}(\mathfrak{C})$ -subflow of $S_p(\mathfrak{C})$. By randomness and q.e., $\Phi_{\upharpoonright X} : X \rightarrow S_{p'}(\mathfrak{C}')$ is an isomorphism. Define μ on Borel subsets of $S_p(\mathfrak{C})$ by $\mu(U) := \nu(\Phi[X \cap U])$. It is clear that μ is an $\text{Aut}(\mathfrak{C})$ -invariant, Borel probability measure on $S_p(\mathfrak{C})$.

The rest of the analysis of Example 5.10 is devoted to the proof that Condition (A') holds for T , i.e. our goal is to show that the canonical Hausdorff quotient of the Ellis group of T is profinite. Since the theory is binary, by Corollary 2.17, we can work with $(\text{Aut}(\mathfrak{C}), S_1(\mathfrak{C}))$ in place of $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$.

Put $E_\infty := \bigwedge_{i \in \omega} E_i$ and $H := 2^\omega$. For each $i \in \omega$, fix an enumeration $C_{i,0}, C_{i,1}$ of the E_i -classes, and an enumeration $(C_\epsilon)_{\epsilon \in H}$ of the E_∞ -classes. Let $X := S_1(\mathfrak{C})$ and put

$$X' := \bigcap_{a, b \in \mathfrak{C}, E_\infty(a, b)} [R(x, a) \leftrightarrow R(x, b)].$$

It is clear that X' is an $\text{Aut}(\mathfrak{C})$ -subflow of X . By q.e., each $p \in X'$ is implied by the union of the following partial types for unique $\epsilon \in H$ and $\delta \in 2^H$:

- $p_\epsilon(x) := \{x \in C_{i, \epsilon(i)} \mid i \in \omega\}$; and
- $q_\delta(x) := \{R(x, a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 1\} \cup \{\neg R(x, a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 0\}$.

Conversely, for each $\epsilon \in H$ and $\delta \in 2^H$ the union of the above partial types implies a type in X' . So X' is topologically identified with the space $H \times 2^H$. Further, for $\text{Stab}(X') := \{\sigma \in \text{Aut}(\mathfrak{C}) \mid \sigma_{\upharpoonright X'} = \text{id}_{X'}\}$, we see that $\text{Aut}(\mathfrak{C})/\text{Stab}(X') \cong H$ and the flow $(\text{Aut}(\mathfrak{C})/\text{Stab}(X'), X')$ can be identified with the flow $(H, H \times 2^H)$ equipped with the following action: For $\sigma \in H$ and $(\epsilon, \delta) \in H \times 2^H$,

$$\sigma(\epsilon, \delta) := (\sigma + \epsilon, \sigma\delta),$$

where $\sigma\delta(\epsilon') := \delta(\epsilon' - \sigma) = \delta(\epsilon' + \sigma)$ for $\epsilon' \in H$.

Claim. *There is $\eta \in \text{EL}(X)$ whose image is contained in X' .*

Proof of Claim. For each $\epsilon \in H$, put $\Delta_\epsilon(x) := \{R(x, a) \leftrightarrow R(x, b) : a, b \in C_\epsilon\}$. First, we will show that for every $\epsilon \in H$ there is $\eta_\epsilon \in \text{EL}(X)$ such that:

- η_ϵ is the limit of a net of $\sigma \in \text{Aut}(\mathfrak{C})$ fixing each $C_{\epsilon'}$, $\epsilon' \in H$, setwise (equivalently, this is a net of Shelah strong automorphism);
- $\text{Im}(\eta_\epsilon) \subseteq [\Delta_\epsilon]$.

For this, choose any representatives $a_{\epsilon'}$ of the classes $C_{\epsilon'}$, $\epsilon' \in H$, and consider any $p_0, \dots, p_{k-1} \in X$ and $a_0, \dots, a_{n-1} \in C_\epsilon$. Since $(C_\epsilon, R|_{C_\epsilon})$ is a (monster) model of the theory of the random graph, by [3, Proposition 3.3] and q.e., there are $a'_0, \dots, a'_n \in C_\epsilon$ such that $\bar{a} := a_0 \dots a_{n-1} \equiv a'_0 \dots a'_{n-1} =: \bar{a}'$ and for each $l < k$:

$$p_l \models \bigwedge_{j, j' < n} R(x, a'_j) \leftrightarrow R(x, a'_{j'}).$$

By randomness, q.e., and saturation, we can find $a'_{\epsilon'} \in C_{\epsilon'}$ for all $\epsilon' \in H$ so that

$$(a_j \mid j < n) \frown (a_{\epsilon'} \mid \epsilon' \in H) \equiv (a'_j \mid j < n) \frown (a'_{\epsilon'} \mid \epsilon' \in H).$$

So, there is $\sigma_{\bar{p}, \bar{a}} \in \text{Aut}(\mathfrak{C})$ mapping the latter sequence above to the former one. Now, it is enough to take η_ϵ to be an accumulation point of the net $(\sigma_{\bar{p}, \bar{a}})_{\bar{p}, \bar{a}}$.

Since the η_ϵ 's are approximated by Shelah strong automorphisms, for each $\epsilon, \epsilon' \in H$, $\eta_\epsilon[[\Delta_{\epsilon'}]] \subseteq [\Delta_{\epsilon'}]$ holds. For $\bar{\epsilon} = \{\epsilon_i\}_{i < n} \subseteq H$ put $\eta_{\bar{\epsilon}} := \eta_{\epsilon_{n-1}} \circ \dots \circ \eta_{\epsilon_0}$, where $\epsilon_0 < \dots < \epsilon_{n-1}$ (say with respect to the lexicographic order). Then, an accumulation point of the net $(\eta_{\bar{\epsilon}})_{\bar{\epsilon}}$ satisfies $\text{Im}(\eta) \subseteq \bigcap [\Delta_\epsilon] = X'$. \square Claim

By the claim and Corollary 2.7, the Ellis groups of the flows $(\text{Aut}(\mathfrak{C}), X)$ and $(\text{Aut}(\mathfrak{C}), X')$ are topologically isomorphic. By the discussion before the claim, the latter Ellis group is topologically isomorphic with the Ellis group of the flow $(H, H \times 2^H)$ which in turn is topologically isomorphic with the Ellis group of the flow $(H, \beta H)$ by Proposition 2.9(ii). As a conclusion, we get that the canonical Hausdorff quotient of the Ellis group of T is topologically isomorphic with the canonical Hausdorff quotient of the Ellis group of βH , i.e. with the generalized Bohr compactification of H . Since H is abelian (and so strongly amenable), Fact 2.10 implies that this generalized Bohr compactification coincides with the Bohr compactification of H , which is profinite by Fact 2.11, as H is abelian of finite exponent. So Condition (A') has been proved.

The next example is a modification of the previous one. It shows that (A'') does not imply (A') in Theorem 3. Moreover, the obtained theory is supersimple of SU-rank 1, so we see that even for supersimple theories the Ellis group need not be profinite, while $\text{Gal}_{KP}(T)$ must be profinite by [2]. Our theory is also amenable.

Example 5.11. The idea is to modify the previous example by considering independent equivalence relations with growing finite number of classes in order to get at the end that the canonical Hausdorff quotient of the Ellis group of the resulting theory is topologically isomorphic with the Bohr compactification of the product $\prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$ which is not profinite by Fact 2.11, as $\prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$ is not of finite exponent.

Consider the relational language $L' = \{R, E_n\}_{2 \leq n < \omega}$, where each symbol binary, and consider an L' -structure M' satisfying the same axioms as M in Example 5.10, except each E_n has exactly n classes. Expand M' to the structure M in the language $L := L' \cup \{O_{n,k}\}_{\substack{2 \leq n < \omega \\ 1 \leq k < n}}$, where each $O_{n,k}$ is a binary relational symbol interpreted as follows: for each $2 \leq n < \omega$ enumerate the E_n -classes as $C_{n,0}, \dots, C_{n,n-1}$ and take

the cyclic permutation $\rho_n := (0, 1, \dots, n-1) \in \text{Sym}(n)$; set $O_{n,k} := \bigcup_{i < n} C_{n,i} \times C_{n,\rho_n^k(i)}$. Let $T := \text{Th}(M)$. A standard back-and-forth argument shows that T has q.e.; in particular, there is only one type in $S_1(T)$. Let $\mathfrak{C} \models T$ be a monster model.

Put $\bar{C}_n := (C_{n,0}, \dots, C_{n,n-1})$ and define $O_n(x_0, \dots, x_n)$ as $\bigwedge_{1 \leq k < n} O_{n,k}(x_0, x_k)$. Observe that $O_n(a_0, \dots, a_{n-1})$ holds iff $(a_0/E_n, \dots, a_{n-1}/E_n)$ belongs to the orbit of \bar{C}_n under the action of $\mathbb{Z}/n\mathbb{Z}$ on $(M/E_n)^n$ defined as follows: for $(b_0, \dots, b_{n-1}) \in (M/E_n)^n$ and $l \in \mathbb{Z}/n\mathbb{Z}$, $l(b_0, \dots, b_{n-1}) := (b_{\rho_n^l(0)}, \dots, b_{\rho_n^l(n-1)})$.

By q.e. and randomness, it easily follows that $\text{tp}(a/A)$ does not fork over \emptyset for any a and A with $a \notin A$. Hence, T is supersimple of SU-rank 1. (The same is true in Example 5.10, but supersimplicity is more important here.) So, by [2], $\text{Gal}_{KP}(T)$ is profinite, i.e. (A'') holds. Amenability of T can be proved as in Example 5.10.

To show that (A') fails, one should follow the lines of the argument in Example 5.10 with $H := \prod_{n \geq 2} \mathbb{Z}/n\mathbb{Z}$. The fact the O_n 's are L -formulae is used to see that $(\text{Aut}(\mathfrak{C})/\text{Stab}(X'), X')$ can be identified with the flow $(H, H \times 2^H)$. At the end, we get that the Ellis group of T is topologically isomorphic with the Bohr compactification of H , which is not profinite by Fact 2.11, i.e. Condition (A') fails.

By [12, Proposition 4.2], if a theory T is extremely amenable, then $\text{Gal}_L(T) = \text{Gal}_{KP}(T)$ is trivial. The next example (whose details are left to the reader) shows that amenability of T does not even imply that $\text{Gal}_{KP}(T)$ is profinite.

Example 5.12. Let $N = (M, X, \cdot)$ be a two-sorted structure, where:

- M is a real closed field in the language $L_{or}(\mathbb{R})$ of ordered rings with constant symbols for all $r \in \mathbb{R}$;
- $\cdot : S^1 \times X \rightarrow X$ is a strictly 1-transitive action of the circle group S^1 on X .

N is clearly interpretable in M , hence $T := \text{Th}(N)$ has NIP. By [6], we easily get that $\text{Gal}_{KP}(T) \cong S^1$, so $\text{Gal}_{KP}(T)$ is not profinite. By [12, Corollary 2.21], we know that a NIP theory is amenable iff \emptyset is an extension base for forking. We leave as a non-difficult exercise to check that in T every set is an extension base.

We have determined the relationships between most of the introduced properties. Let us discuss a few remaining questions.

Question 5.13. Is there an example for which (C) holds but (D) does not?

Question 5.14. (i) Is there an example for which (A) holds but (C) does not?

(ii) Is there an example for which (A') holds but (A) does not?

Example 5.10 shows that (A') does not imply (B). So this example either witnesses that (A') does not imply (A), or that (A) does not imply (B); but we do not know which of these two lack of implications is witnessed by Example 5.10. By Remark 2.2(a), it witnesses the lack of the implication (A') \implies (A) if and only if the Ellis group of the theory in this example is not Hausdorff.

By Example 5.1, we know that having sep. fin. $EDEERdeg$ does not imply $DEERP$; this is witnessed by any stable theory with a non-stationary type in $S_1(\emptyset)$. Is the converse true, i.e. does $DEERP$ (equiv. extreme amenability) imply sep. fin. $EDEERdeg$? Probably not. Example 5.10 shows that $DEERP$ (equiv. amenability) does not imply sep. fin. $EDEERdeg$. In fact, Example 5.12 shows that amenability does not even imply that Gal_{KP} is profinite (i.e. (A'')), whereas extreme amenability implies that Gal_{KP} is trivial by [12, Proposition 4.2].

By Example 5.5(e), we know that the Ellis group there is profinite. Is it infinite? Can one compute it precisely, as we did for the theory from Example 5.7? The

same problem for all random hypergraphs, although here we know that the Ellis groups are finite by the paragraph after Example 5.5.

APPENDIX A. EXAMPLE 5.7

We now calculate the Ellis group from Example 5.7. Recall that we consider the language $L := \{R_2, R_4\}$, where R_2 is a binary and R_4 is a quaternary relational symbol. We consider the class of all finite $(2, 4)$ -hypergraphs, i.e. the class of all finite structures in L for which R_2 and R_4 are irreflexive and symmetric. This is a Fraïssé class, and its Fraïssé limit K is called the random $(2, 4)$ -hypergraph. The theory $T := \text{Th}(K)$ is \aleph_0 -categorical, and so \aleph_0 -saturated with q.e.

We also consider the following expansion of T . Let $L^* = L \cup \{<\}$. Consider the class of all finite linearly ordered $(2, 4)$ -hypergraphs. This is a Fraïssé class, and its Fraïssé limit K^* is called the ordered random $(2, 4)$ -hypergraph. The theory $T^* := \text{Th}(K^*)$ is \aleph_0 -categorical, so \aleph_0 -saturated with q.e. Then $K_{\uparrow L}^* \cong K$, so $T_{\uparrow L}^* = T$, and we may assume that $K_{\uparrow L}^* = K$.

By a straightforward back-and-forth argument one can easily see that there exists $\rho \in \text{Aut}(K)$ such that ρ reverses the order on K^* : $\rho[<] = >$. Take $L_\rho^* := L^* \cup \{\rho\}$ and the obvious expansion K_ρ^* of K^* . Take a monster \mathfrak{C}_ρ^* of $\text{Th}(K_\rho^*)$ such that $\mathfrak{C}^* := \mathfrak{C}_\rho^* \upharpoonright L^*$ and $\mathfrak{C} := \mathfrak{C}_\rho^* \upharpoonright L$ are monster models of T^* and T , respectively. The interpretation of ρ in \mathfrak{C}_ρ^* , which will be also denoted by ρ , is an automorphism of \mathfrak{C} reversing $<$. Further on, we fix \mathfrak{C}^* , \mathfrak{C} , and ρ .

Since T^* has *EDEERP* (even *EERP* by Example 5.4(a)), by Theorem 3.14, we can find $u^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ with $\text{Im}(u^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$. By Lemma 2.20(iv), for a single variable z , we can find $u \in \text{EL}(S_z(\mathfrak{C}))$ such that $\text{Im}(u) \subseteq \text{Inv}_z^*(\mathfrak{C})$, where $\text{Inv}_z^*(\mathfrak{C})$ is the set of all $\text{Aut}(\mathfrak{C}^*)$ -invariant types in $S_z(\mathfrak{C})$. Moreover, we may assume that $u \in \mathcal{J}(\mathcal{M})$ for a minimal left ideal \mathcal{M} of $\text{EL}(S_z(\mathfrak{C}))$.

The elements of $\text{Inv}_z^*(\mathfrak{C})$ are not hard to describe. Let $p(z) \in \text{Inv}_z^*(\mathfrak{C})$. Since there is only one type in $S_z(T^*)$, $p(z)$ either contains $R_2(z, a)$ for all $a \in \mathfrak{C}$ or $\neg R_2(z, a)$ for all $a \in \mathfrak{C}$. Note that $S_{\pi(\bar{y})}(T^*)$, where $\bar{y} = (y_0, y_1, y_2)$ and $\pi(\bar{y}) = \{y_0 \neq y_1 \neq y_2 \neq y_0\}$, is completely determined by restriction to $\{R_2, <\}$. Let us write $[\mathfrak{C}]^3 = O_0 \sqcup O_1 \sqcup O_2 \sqcup O_3$, where O_i is the set of all $\{a, b, c\} \in [\mathfrak{C}]^3$ with exactly i -many R_2 -edges on the set $\{a, b, c\}$. Note that by symmetry of R_4 , either $R_4(z, a, b, c) \in p(z)$ for all $\{a, b, c\} \in O_0$, or $\neg R_4(z, a, b, c) \in p(z)$ for all $\{a, b, c\} \in O_0$. The same holds for O_3 . The sets O_1 and O_2 are more interesting. Write $O_1 = O_1^- \sqcup O_1^\circ \sqcup O_1^+$, where for $\{a, b, c\} \in O_1$ with b being R_2 -unconnected with a and c we put:

$$\{a, b, c\} \in \begin{cases} O_1^- & \text{if } b \text{ is minimal among } \{a, b, c\} \\ O_1^\circ & \text{if } b \text{ is the middle one among } \{a, b, c\} \\ O_1^+ & \text{if } b \text{ is maximal among } \{a, b, c\} \end{cases} .$$

Similarly, we write $O_2 = O_2^- \sqcup O_2^\circ \sqcup O_2^+$, where the division is determined by the element R_2 -connected to both other elements. By symmetry of R_4 , we have either $R_4(z, a, b, c) \in p(z)$ for all $\{a, b, c\} \in O_1^-$, or $\neg R_4(z, a, b, c) \in p(z)$ for all $\{a, b, c\} \in O_1^-$. The same holds for $O_1^\circ, O_1^+, O_2^-, O_2^\circ$ and O_2^+ . By q.e., $p(z)$ is completely determined by the previous information. Moreover, by randomness, each described possibility occurs. Thus, we see that $\text{Inv}_z^*(\mathfrak{C})$ has 2^9 elements.

Lemma A.1. $u(p) = p$ for all $p \in \text{Inv}_z^*(\mathfrak{C})$. In particular, $\text{Im}(u) = \text{Inv}_z^*(\mathfrak{C})$.

Proof. It is enough to prove the first part. If $R_2(z, a)^\epsilon \in p$ for $\epsilon \in 2$ and $a \in \mathfrak{C}$, then $R_2(z, a)^\epsilon \in \sigma(p)$ for every $\sigma \in \text{Aut}(\mathfrak{C})$, so $R_2(z, a)^\epsilon \in u(p)$ holds as well. Similarly, if $R_4(z, a, b, c)^\epsilon \in p$ for $\epsilon \in 2$ and $\{a, b, c\} \in O_0$, then $R_4(z, a, b, c)^\epsilon \in \sigma(p)$ for every $\sigma \in \text{Aut}(\mathfrak{C})$, so $R_4(z, a, b, c)^\epsilon \in u(p)$. The same holds for O_3 .

Note that if we put $q = u(p)$, then, by idempotency, $u(q) = u^2(p) = u(p) = q$.

Let us focus on O_1 . We say that $p \in \text{Inv}_z^*(\mathfrak{C})$ is of type $(\epsilon_-, \epsilon_o, \epsilon_+)$, where $\epsilon_-, \epsilon_o, \epsilon_+ \in 2$, if $R_4^{\epsilon_\star}(z, a, b, c) \in p$ for all $\{a, b, c\} \in O_1^\star$, for each $\star \in \{-, o, +\}$.

Claim. p and $q = u(p)$ have the same type.

Proof of Claim. If p is of type $(0, 0, 0)$ or $(1, 1, 1)$, then q is of the same type, as all automorphisms in these cases preserve the type of p , and hence u preserves it, too. Take a, b, c, d such that $a < b < c < d$, $R_2(a, d)$, $R_2(b, c)$, and there are no other R_2 -edges between a, b, c, d . Let q be of type $(\epsilon_-, \epsilon_o, \epsilon_+)$. Then the formula $\phi(z, a, b, c, d) :=$

$$R_4^{\epsilon_-}(z, a, b, c) \wedge R_4^{\epsilon_o}(z, b, a, d) \wedge R_4^{\epsilon_o}(z, c, a, d) \wedge R_4^{\epsilon_+}(z, d, b, c) \in q = u(p) = u(q),$$

We can approximate u by $\sigma \in \text{Aut}(\mathfrak{C})$ such that $\phi(z, a, b, c, d) \in \sigma(p), \sigma(q)$, i.e. $\phi(z, a', b', c', d') \in p, q$, where $\sigma(a', b', c', d') = (a, b, c, d)$.

To see that p and q are of the same type, we do the following case analysis. E.g. if q is of type $(1, 0, 0)$, then p is of one of the types $(1, 0, 0)$, $(1, 1, 0)$ or $(1, 0, 1)$, as $R_4(z, a', b', c') \in q$ implies $a' < b', c'$, so $R_4(z, a', b', c') \in p$. If p is of type $(1, 1, 0)$, then $\neg R_4(z, b', a', d') \wedge \neg R_4(z, d', b', c') \in p$ implies $b' > a', d'$ and $d' > b', c'$ which is not possible. Similarly, one eliminates that p is of type $(1, 0, 1)$.

A similar analysis works when q is of type $(0, 0, 1)$ and $(0, 1, 0)$, and also shows that p is not of type $(1, -, -)$ nor $(-, -, 1)$ when q is of type $(0, 0, 0)$. Exceptionally, to eliminate the case when q is of type $(0, 0, 0)$ and p is of type $(0, 1, 0)$ a different trick is required.

Choose elements a_0, \dots, a_4 such that $R_2(a_i, a_{i+1})$ for all $i < 5$ (+ is modulo 5), and there are no other R_2 -edges between them. Then $\bigwedge_{i < 5} \neg R_4(z, a_i, a_{i+2}, a_{i+3}) \in q = u(p) = u(q)$, as q is of type $(0, 0, 0)$. As above, we can approximate u by σ and find a copy $(a'_0, a'_1, a'_2, a'_3, a'_4)$ of $(a_0, a_1, a_2, a_3, a_4)$ such that both p and q contain $\bigwedge_{i < 5} \neg R_4(z, a'_i, a'_{i+2}, a'_{i+3})$. Since p is of type $(0, 1, 0)$, we get that $a'_i < a'_{i+2}, a'_{i+3}$ or $a'_i > a'_{i+2}, a'_{i+3}$, for all $i < 5$. But it is easy to see that this is impossible (just looking at $<$).

The remaining cases are completely dual by interchanging 0 and 1, and R_4 and $\neg R_4$ in the previous cases. □ Claim

The above analysis for O_1 applies to O_2 by interchanging all R_2 -edges and R_2 -non-edges. The lemma is proved. □

We can now easily see that $u\mathcal{M}$ is not trivial. Take $p, q \in \text{Inv}_z^*(\mathfrak{C})$ such that $p(z)$ implies that z is not R_2 -connected to anything and only R_4 -connected to O_1^- , and $q(z)$ implies that z is not R_2 -connected to anything and only R_4 -connected to O_1^+ . Note that $u\rho u \in u\mathcal{M}$ and that $\rho[O_1^-] = O_1^+$ and vice versa. Thus, by Lemma A.1, $u\rho u(p) = u\rho(p) = u(q) = q$, so $u\rho u \neq u$ as $u(p) = p$. We will see that $u\mathcal{M} = \{u, u\rho u\}$, but this will require more work, involving applications of contents.

Lemma A.2. $u\mathcal{M} = \{u, u\rho u\}$, so $u\mathcal{M} \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. Since $\text{Im}(u) \subseteq \text{Inv}_z^*(\mathfrak{C})$, it is enough to prove that for any $\eta \in u\mathcal{M}$: either $\eta(p) = u(p)$ for all $p \in \text{Inv}_z^*(\mathfrak{C})$, or $\eta(p) = u\rho u(p)$ for all $p \in \text{Inv}_z^*(\mathfrak{C})$. We define the

notion of O_1 -type and O_2 -type of $p \in \text{Inv}_z^*(\mathfrak{C})$ as in the proof of Lemma A.1. Fix $\eta \in u\mathcal{M}$; p will always range over $\text{Inv}_z^*(\mathfrak{C})$.

Claim. *If the O_1 -type of p is $(\epsilon_-, \epsilon_o, \epsilon_+)$, then the O_1 -type of $\eta(p)$ is $(\epsilon_-, \epsilon_o, \epsilon_+)$ or $(\epsilon_+, \epsilon_o, \epsilon_-)$. The same holds for O_2 -types.*

Proof of Claim. Let us first deal with O_1 -types. If $\epsilon_- = \epsilon_o = \epsilon_+$, then each automorphism preserves the O_1 -type of p , hence η preserves it, too, and we are done.

Take elements a_0, a_1, a_2, a_3 such that $R_2(a_0, a_2)$ and $R_2(a_1, a_3)$, and there are no other R_2 -edges between them. Put $q(y_0, y_1, y_2, y_3) := \text{tp}^L(a_0, a_1, a_2, a_3)$. For a realization \bar{b} of q we will say that it is of type:

- A: if $\min(\bar{b})$ is R_2 -connected to $\max(\bar{b})$ (min and max are taken in \mathfrak{C}^*);
- B: if $\min(\bar{b})$ and $\max(\bar{b})$ are not R_2 -connected and $\min(\bar{b})^* < \max(\bar{b})^*$, where $b_i^* = b_{i+2}$, so it is the element to which b_i is R_2 -connected;
- C: if $\min(\bar{b})$ and $\max(\bar{b})$ are not R_2 -connected and $\max(\bar{b})^* < \min(\bar{b})^*$.

For a type p , let $\delta_0, \delta_1, \delta_2, \delta_3 \in \{0, 1\}$ be the unique numbers such that the formula $\bigwedge_{i < 4} R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2})$ belongs to p . Note that they depend on p and $\bar{b} \models q$. Denote this formula by $\phi_{p, \bar{b}}(z, \bar{b})$. In the following table, we calculate $\sum_{i < 4} \delta_i$ depending on the O_1 -type of p and the type of ordering on \bar{b} :

	(0,0,0)	(1,0,0)	(0,1,0)	(0,0,1)	(1,1,0)	(1,0,1)	(0,1,1)	(1,1,1)
A/C	0	1	2	1	3	2	3	4
B	0	2	0	2	2	4	2	4

Recall that by Fact 2.23, $\text{ct}(\eta(p)) \subseteq \text{ct}(p)$. By our choice, $(\phi_{p, \bar{b}}(z, \bar{y}), q(\bar{y})) \in \text{ct}(p)$ for every $\bar{b} \models q$. Consider the following cases.

Case 1. p is of O_1 -type $(1, 0, 0)$ or $(0, 0, 1)$. If $\eta(p)$ is of type $(0, 0, 0)$, $(0, 1, 0)$, $(1, 0, 1)$ or $(1, 1, 1)$, then choose $\bar{b} \models q$ of type B. Since $(\phi_{\eta(p), \bar{b}}(z, \bar{y}), q(\bar{y})) \in \text{ct}(\eta(p))$, this pair belongs to $\text{ct}(p)$ as well. But, by the table, this is not possible, since $\phi_{\eta(p), \bar{b}}$ has either 0 or 4 positive occurrences of R_4 , whereas this does not happen in $\phi_{p, \bar{b}'}$ for any $\bar{b}' \models q$ if p is of type $(1, 0, 0)$ or $(0, 0, 1)$. Similarly, if $\eta(p)$ is of type $(1, 1, 0)$ or $(0, 1, 1)$, by choosing $\bar{b} \models q$ of type A, we have $(\phi_{\eta(p), \bar{b}}(z, \bar{y}), q(\bar{y})) \in \text{ct}(\eta(p)) \subseteq \text{ct}(p)$, but since $\phi_{\eta(p), \bar{b}}$ has 3 positive occurrences of R_4 , we again cannot find $\bar{b}' \models q$ such that $\phi_{\eta(p), \bar{b}}(z, \bar{b}') \in p$. So, $\eta(p)$ is either of O_1 -type $(1, 0, 0)$ or $(0, 0, 1)$.

Case 2. p is of O_1 -type $(0, 1, 0)$. By similar considerations to those in Case 1, we obtain that the only possibilities for the O_1 -type of $\eta(p)$ are $(0, 0, 0)$ and $(0, 1, 0)$. The case when $\eta(p)$ is of O_1 -type $(0, 0, 0)$ requires a different trick, but this can be done in the same way as in the proof of Lemma A.1. So $\eta(p)$ is of O_1 -type $(0, 1, 0)$.

The remaining cases are dual. Interchanging R_2 edges and R_2 -non-edges, we obtain the claim for O_2 -types. \square Claim

Note that if p is of O_1 -type $(\epsilon_-, \epsilon_o, \epsilon_+)$, then $\rho(p)$ is of O_1 -type $(\epsilon_+, \epsilon_o, \epsilon_-)$, and similarly for O_2 -types. Therefore, the previous claim says that $\eta(p)$ has the same O_1 -type [O_2 -type] as p or as $\rho(p)$.

Claim. *Either for every p the O_1 -types of p and $\eta(p)$ are equal, or for every p the O_1 -types of $\rho(p)$ and $\eta(p)$ are equal. The same holds for O_2 -types.*

Proof of Claim. Consider the case of O_1 -types (the case of O_2 -types follows by interchanging all R_2 -edges and R_2 -non-edges). Suppose not. Then we have types

p, p' with O_1 -types $(\epsilon_-, \epsilon_o, \epsilon_+)$ and $(\epsilon'_-, \epsilon'_o, \epsilon'_+)$ such that $\eta(p)$ and $\eta(p')$ are of O_1 -types $(\epsilon_-, \epsilon_o, \epsilon_+)$ and $(\epsilon'_+, \epsilon'_o, \epsilon'_-)$, respectively, where $\epsilon_- \neq \epsilon_+$ and $\epsilon'_- \neq \epsilon'_+$. We have two cases.

Case 1. $(\epsilon_-, \epsilon_+) = (\epsilon'_-, \epsilon'_+)$. Consider $a_i, i < 4$, such that $R_2(a_0, a_2), R_2(a_1, a_3)$, and there are no other R_2 -edges between them, and $a_0, a_2 < a_1, a_3$. Let $q(\bar{y}) = \text{tp}^L(a_0, a_1, a_2, a_3)$. Choose $\delta_0, \delta_1, \delta_2, \delta_3$ such that $\bigwedge_{i < 4} R_4^{\delta_i}(z, a_i, a_{i+1}, a_{i+2}) \in \eta(p)$. Then $\bigwedge_{i < 4} R_4^{1-\delta_i}(z, a_i, a_{i+1}, a_{i+2}) \in \eta(p')$, so:

$$\left(\bigwedge_{i < 4} R_4^{\delta_i}(z, y_i, y_{i+1}, y_{i+2}), \bigwedge_{i < 4} R_4^{1-\delta_i}(z, y_i, y_{i+1}, y_{i+2}), q(\bar{y}) \right) \in \text{ct}(\eta(p), \eta(p')).$$

By Fact 2.23, this triple belongs to $\text{ct}(p, p')$, so $\bigwedge_{i < 4} R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p$ and $\bigwedge_{i < 4} R_4^{1-\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p'$ for some $\bar{b} \models q$. Choose i such that $b_{i+1} = \min(\bar{b})$. Then $R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p$ implies that $\epsilon_- = \delta_i$, and $R_4^{1-\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p'$ implies that $\epsilon'_- = 1 - \delta_i$. Therefore, $\epsilon_- \neq \epsilon'_-$; a contradiction.

Case 2. $(\epsilon_-, \epsilon_+) \neq (\epsilon'_-, \epsilon'_+)$. Then $(\epsilon_-, \epsilon_+) = (\epsilon'_+, \epsilon'_-)$, so we reduce this to Case 1 by considering $\eta(p)$ and $\eta(p')$ instead of p and p' , and η^{-1} (computed in $u\mathcal{M}$) instead of η (note that $\eta^{-1}(\eta(p)) = u(p) = p$ and $\eta^{-1}(\eta(p')) = u(p') = p'$ by Lemma A.1). \square Claim

We finally prove:

Claim. *Either for every p the O_1 -types of p and $\eta(p)$ are equal and the O_2 -types of p and $\eta(p)$ are equal, or for every p the O_1 -types of $\rho(p)$ and $\eta(p)$ are equal and the O_2 -types of $\rho(p)$ and $\eta(p)$ are equal.*

Proof of Claim. Let p have both the O_1 -type and the O_2 -type equal to $(1, 0, 0)$. If the claim fails, then, by the previous two claims, the O_1 -types of p and $\eta(p)$ are equal and the O_2 -types of $\rho(p)$ and $\eta(p)$ are equal, or the O_1 -types of $\rho(p)$ and $\eta(p)$ are equal and the O_2 -types of p and $\eta(p)$ are equal.

So, assume first that $\eta(p)$ has O_1 -type $(1, 0, 0)$ but O_2 -type $(0, 0, 1)$. Consider a_0, a_1, a_2, a_3 such that $R_2(a_0, a_1), R_2(a_0, a_3)$ and there are no other R_2 -edges between them, and $a_0 > a_1 > a_2 > a_3$; set $q(\bar{y}) = \text{tp}^L(a_0, a_1, a_2, a_3)$. Then $R_4(z, a_2, a_0, a_1) \wedge \neg R_4(z, a_2, a_0, a_3) \wedge R_4(z, a_0, a_1, a_3) \in \eta(p)$. But, by Fact 2.23, $\text{ct}(\eta(p)) \subseteq \text{ct}(p)$. Hence, we can find $\bar{b} \models q$ such that $R_4(z, b_2, b_0, b_1) \wedge \neg R_4(z, b_2, b_0, b_3) \wedge R_4(z, b_0, b_1, b_3) \in p$. Since the O_1 -type and the O_2 -type of p are both $(1, 0, 0)$, this implies $b_2 < b_0, b_1, b_0 < b_1, b_3$, but b_2 is not less than both b_0 and b_3 . Clearly, this is not possible.

If $\eta(p)$ has O_1 -type $(0, 0, 1)$ but O_2 -type $(1, 0, 0)$, the proof is dual by reversing the order on $\{a_0, a_1, a_2, a_3\}$. \square Claim

We are ready to finish the proof of the lemma. If p contains $R_2^\epsilon(z, a)$ for some $\epsilon \in 2$ and all $a \in \mathcal{C}$, then $\sigma(p)$ contains it, too, and so does $\eta(p)$. Similarly, if p contains $R_4^\epsilon(z, a, b, c)$ for some $\epsilon \in 2$ and all $\{a, b, c\} \in O_0$ [resp. $\in O_3$], then $\eta(p)$ contains it, too. Thus, the restrictions of $\eta(p), p$, and $\rho(p)$ to these formulae coincide for every $p \in \text{Inv}_z^*(\mathcal{C})$. By the previous claim, either for every $p \in \text{Inv}_z^*(\mathcal{C})$ the restrictions of $\eta(p)$ and p to the formulae $R_4^\epsilon(z, a, b, c)$ for $\epsilon \in 2$ and $\{a, b, c\} \in O_1 \cup O_2$ coincide, or for every $p \in \text{Inv}_z^*(\mathcal{C})$ the restrictions of $\eta(p)$ and $\rho(p)$ to these formulae coincide. Therefore, either for every $p \in \text{Inv}_z^*(\mathcal{C})$ we have $\eta(p) = p = u(p)$, or for every $p \in \text{Inv}_z^*(\mathcal{C})$ we have $\eta(p) = \rho(p) = u\rho(p)$. But this means that either $\eta = u$, or $\eta = u\rho$. \square

Proposition A.3. *The Ellis group of $(\text{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ is $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Take $u^* \in \text{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\text{Im}(u^*) \subseteq \text{Inv}_{\bar{c}}(\mathfrak{C}^*)$, as T^* has *EDEERP*. By Corollary 2.20(ii), there is $u' \in \text{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\text{Im}(u') \subseteq \text{Inv}_{\bar{c}}^*(\mathfrak{C})$. Furthermore, we may assume that u' is an idempotent in a minimal left ideal \mathcal{M}' of $\text{EL}(S_{\bar{c}}(\mathfrak{C}))$. By Lemma 2.14 (having in mind the natural identification of $S_{\bar{d}}(\mathfrak{C})$ with $S_{\bar{c}}(\mathfrak{C})$), we have the flow and semigroup epimorphism $\Phi : \text{EL}(S_{\bar{c}}(\mathfrak{C})) \rightarrow \text{EL}(S_z(\mathfrak{C}))$ (where z is a single variable) given by:

$$\Phi(\eta)(p(z)) = \hat{\eta}(p(z)) := \eta(q(\bar{x}))_{\uparrow x'}[x'/z],$$

where $p(z) \in S_z(\mathfrak{C})$, $x' \in \bar{x}$, and $q(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$ are such that $q(\bar{x})_{\uparrow x'}[x'/z] = p(z)$. By Fact 2.3, $u := \Phi(u')$ is an idempotent in the minimal left ideal $\mathcal{M} := \Phi[\mathcal{M}']$ of $\text{EL}(S_z(\mathfrak{C}))$, and $\Phi_{\uparrow u' \mathcal{M}'} : u' \mathcal{M}' \rightarrow u \mathcal{M}$ is a group epimorphism. By the formula above, $\text{Im}(u) \subseteq \text{Inv}_z^*(\mathfrak{C})$, so by Lemma A.1 and Lemma A.2, we have that $\text{Im}(u) = \text{Inv}_z^*(\mathfrak{C})$ and $u \mathcal{M} = \{u, u \rho u\}$ has two elements. So it remains to show that $\ker(\Phi_{\uparrow u' \mathcal{M}'})$ is trivial.

Let $\eta \in u' \mathcal{M}'$ be such that $\hat{\eta} = u$. It is enough to prove that $\eta(q) = q$ for all $q \in \text{Im}(\eta) = \text{Im}(u')$ (recall that all such q 's are $\text{Aut}(\mathfrak{C}^*)$ -invariant).

If $R_2^c(x_i, a) \in \eta(q)$, then $R_2^c(x_i, \sigma(a)) \in q$ for some $\sigma \in \text{Aut}(\mathfrak{C})$, but then $R_2^c(x_i, a) \in q$ by $\text{Aut}(\mathfrak{C}^*)$ -invariance of q (as there is only one type in $S_1(T^*)$). Similarly, if $R_4^c(x_i, x_j, x_k, a) \in \eta(q)$, then $R_4^c(x_i, x_j, x_k, a) \in q$ by invariance. If $R_4^c(x_i, x_j, a, b) \in \eta(q)$, by symmetry of R_4 and invariance of q , the conclusion is the same: $R_4^c(x_i, x_j, a, b) \in q$. Let us consider $R_4^c(x_i, a, b, c) \in \eta(q)$. Let $p(z) = q(\bar{x})_{\uparrow x_i}[x_i/z]$; note that $p(z) \in \text{Inv}_z^*(\mathfrak{C})$, so $p(z) \in \text{Im}(u)$. Then $R_4^c(z, a, b, c) \in \hat{\eta}(p) = u(p) = p = q_{\uparrow x_i}[x_i/z]$. Thus $R_4^c(x_i, a, b, c) \in q$.

By q.e., we conclude that $\eta(q) \subseteq q$, so $\eta(q) = q$, and we are done. \square

ACKNOWLEDGMENTS

The first author would like to thank Pierre Simon for inspiring discussions on random hypergraphs. The authors are also grateful to Tomasz Rzepecki for some useful suggestions which helped us to complete our analysis of Example 5.7 and for suggesting Example 5.8.

REFERENCES

- [1] Auslander, J.: Minimal flows and their extensions. Mathematics Studies 153, North-Holland, The Netherlands (1988)
- [2] Buechler, S., Pillay, A., Wagner, W.: Supersimple theories. J. Amer. Math. Soc. (1) **14**, 109-124 (2001)
- [3] Cameron, P. J.: The random graph. In: The Mathematics of Paul Erdős II, Springer, New York, 353-378 (2013)
- [4] Casanovas, E., Lascar, D., Pillay, A., Ziegler, M.: Galois groups of first order theories. J. Math. Log. **1**, 305-319 (2001)
- [5] Gismatullin, J., Jagiella, G., Krupiński, K.: Bohr compactifications of groups and rings, in preparation (2019)
- [6] Gismatullin, J., Newelski, L.: G-compactness and groups. Arch. Math. Logic **47**, 479-501 (2008)
- [7] Glasner, S.: Proximal Flows. LNM 517, Springer, Germany (1976)
- [8] Glasner, E.: Ergodic Theory via Joinings. Math. Surveys and Monographs 101, AMS, Providence (2003)
- [9] Graham, R. L., Leeb, K., Rothschild, B. L.: Ramsey's theorem for a class of categories. Proc. Natl. Acad. Sci. USA (1) **69**, 119-120 (1972)

- [10] Graham, R. L., Rothschild, B. L.: Ramsey's theorem for n -parameter sets. *Trans. Amer. Math. Soc.* **159**, 257-292 (1971)
- [11] Hrushovski, E.: Definability patterns and their symmetries. arXiv: 1911.01129 (2019)
- [12] Hrushovski, E., Krupiński, K., Pillay, A.: On first order amenability. arXiv:2004.08306v2 (2021)
- [13] Jasiński, J., Laflamme, C., Nguyen Van Thé, L., Woodrow, R.: Ramsey precompact expansions of homogeneous directed graphs. arXiv: 1310.6466 (2013)
- [14] Kechris, A. S., Pestov, V. G., Todorčević, S.: Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. *Geom. Funct. Anal.* (1) **15**, 106-189 (2005)
- [15] Krupiński, K., Newelski, L., Simon, P.: Boundedness and absoluteness of some dynamical invariants in model theory. *J. Math. Log.* (2) **19**, 1950012, 55 pages (2019)
- [16] Krupiński, K., Pillay, A.: Generalized Bohr compactification and model-theoretic connected components. *Math. Proc. Cambridge Philos. Soc.* **163**, 219-249 (2017)
- [17] Krupiński, K., Pillay, A.: On the topological dynamics of automorphism groups; a model-theoretic perspective. arXiv: 1909.13285 (2019)
- [18] Krupiński, K., Pillay, A., Rzepecki, T.: Topological dynamics and the complexity of strong types. *Israel J. Math.* (2) **228**, 863-932 (2018)
- [19] Krupiński, K., Rzepecki, T.: Galois groups as quotients of Polish groups. *J. Math. Log.*, to appear. arXiv: 1804.09247 (2018)
- [20] Moore, J.: Amenability and Ramsey theory. *Fund. Math.* **220**, 263-280 (2013)
- [21] Nešetřil, J., Rödl, V.: Ramsey classes of set systems. *J. Combin. Theory Ser. A* (2) **34**, 183-201 (1983)
- [22] Newelski, L.: Topological dynamics of definable groups actions. *J. Symbolic Logic* **74**, 50-72 (2009)
- [23] Nguyen Van Thé, L.: More on the Kechris-Pestov-Todorčević correspondence: precompact expansions. *Fund. Math.* **222**, 19-47 (2013)
- [24] Nguyen Van Thé, L.: Fixed points in compactifications and combinatorial counterparts. *Ann. Henri Lebesgue* **2**, 149-185 (2019)
- [25] Rzepecki, T.: Bounded Invariant Equivalence Relations. PhD Thesis, Wrocław, arXiv: 1810.05113 (2018)
- [26] Salles, T. S.: Extreme Amenability of Topological Groups. Master's Thesis, Bonn (2011)
- [27] Sokić, M.: Ramsey properties of finite posets. *Order* (1) **29**, 1-30 (2012)
- [28] Sokić, M.: Ramsey property, ultrametric spaces, finite posets, and universal minimal flows. *Israel J. Math.* (2) **194**, 609-640 (2013)
- [29] Turek, S.: Minimal actions on Cantor cubes. *Bull. Pol. Acad. Sci. Math.* (2) **51**, 129-138 (2003)
- [30] Ziegler, M.: Introduction to the Lascar group. In: *Tits buildings and the model theory of groups* (Würzburg, 2000), London Math. Soc. Lecture Note Ser. 291, Cambridge University Press, Cambridge, 279-298 (2002)
- [31] Zucker, A.: Topological dynamics of automorphism groups, ultrafilter combinatorics, and the Generic Point Problem. *Trans. Amer. Math. Soc.* (9) **368**, 6715-6740 (2016)

(K. Krupiński) UNIWERSYTET WROCŁAWSKI, INSTYTUT MATEMATYCZNY, PLAC GRUNWALDZKI 2/4, 50-384 WROCŁAW, POLAND
Email address: `kkrup@math.uni.wroc.pl`

(J. Lee) KAIST, DEPARTMENT OF MATHEMATICAL SCIENCES, 291 DAEHAK-RO YUSEONG-GU, 34141, DAEJEON, SOUTH KOREA
Email address: `ljwhayo@kaist.ac.kr`

(S. Mocomja) UNIVERSITY OF BELGRADE, FACULTY OF MATHEMATICS, STUDENTSKI TRG 16, 11000 BELGRADE, SERBIA
Email address: `slavko@matf.bg.ac.rs`