Stable groups, List 1

We work in a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of an ω-stable theory T. G denotes a group $\emptyset$-definable in $\mathfrak{C}$.

Problem 1. Prove that G^0 is a normal, $\emptyset$-definable subgroup of G.

Problem 2. Prove that the following conditions are equivalent.
(i) $\mathrm{tp}(a/A)$ is a generic type of G.
(ii) For every $b \in G$, if $a \mathop{\smile\!\!\!\!\!\!|}_A b$, then $b \cdot a \mathop{\smile\!\!\!\!\!\!|}_A b$.
(iii) For every $b \in G$, if $a \mathop{\smile\!\!\!\!\!\!|}_A b$, then $b \cdot a \mathop{\smile\!\!\!\!\!\!|} A, b$.

Problem 3. Let $p \in S_G(\mathfrak{C})$. Prove that p is generic iff for every $g \in G$, gp does not fork over $\emptyset$.

Problem 4. (i) For a stationary type $p \in S_G(A)$ and $g \in G$ define $g \cdot p := \mathrm{tp}(g \cdot a/A)$, where $a \models p|A, g$. Check that this definition is correct, i.e. does not depend on the choice of a.
(ii) For a global type $q \in S_G(\mathfrak{C})$ define $g \cdot q := \mathrm{tp}(g \cdot a/\mathfrak{C})$, where $a \models q$. Note that if $p \in S_G(A)$ is stationary, then $g \cdot p = (g \cdot \widetilde{p})|A$.
(iii) Conclude that if $p \in S_G(A)$ is stationary, then $\mathrm{Stab}(p) = \mathrm{Stab}(\widetilde{p})$, where $\mathrm{Stab}(p) := \{g \in G : g \cdot p = p\}$ and $\mathrm{Stab}(\widetilde{p}) := \{g \in G : g \cdot \widetilde{p} = \widetilde{p}\}$.
(iv) Show that the above $\cdot : G \times S_G(\mathfrak{C}) \to S_G(\mathfrak{C})$ is an action of G on $S_G(\mathfrak{C})$. Explain why the above $\cdot : G \times S_G(\mathrm{acl}(\emptyset)) \to S_G(\mathrm{acl}(\emptyset))$ is not an action.

Problem 5. Let H be an A-definable subgroup of G, cH its A-definable coset, and $a \in X$. Prove that $\mathrm{tp}(a/A)$ is a generic type of X iff there is $b \in X$ such that [$a \mathop{\smile\!\!\!\!\!\!|}_A b$ and] $b^{-1} \cdot a$ is generic in H over A, b.

Problem 6. Let $p \in S_G(A)$ be stationary. Prove that:
(i) $\mathrm{Stab}(p)$ is an A-definable subgroup of G^0,
(ii) $\mathrm{Stab}(p) = G^0$ iff p is a generic type of G,
(iii) if $p \subseteq_{nf} q$, then $\mathrm{Stab}(p) = \mathrm{Stab}(q)$.

Problem 7. Let $p = \mathrm{tp}(a/A) \in S_G(A)$ be stationary and let $H := \mathrm{Stab}(p)$. Assume that $X := H \cdot a$ is $\mathrm{acl}(A)$-definable. In the lecture, it was shown that then X is A-definable and p is a generic type of X. Prove that H is connected, i.e. $H = H^0$.

Problem 8. The main corollary of the second lecture says that TFAE:
(i) G is 1-based,
(ii) for every $n \in \omega$, any definable subset of G^n is a Boolean combination of $\mathrm{acl}(\emptyset)$-definable subgroups of G^n.

Drop the assumption that T is stable, and assume that $\mathfrak{C} = (G, \cdot, \dots)$. Prove that property (ii) implies that T is stable.

Stable groups, List 1'

The problems on this auxiliary list are not meant to be solved during the discussion sessions. Regarding Problems 1, 2, and 3, students who did not attend stability theory course should solve them by themselves. Problem 4 belongs to the main stream of this course, but it is harder.

We work in a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of a stable theory.

Problem 1. Work in a monster model of any theory. Let $b \in \operatorname{acl}(Aa)$. Prove that $\mathrm{RM}(a/A) \geq \mathrm{RM}(b/A)$.

Problem 2. Let $p \in S(A)$ be stationary and $B \subseteq \mathfrak{C}$. Show that ($\tilde{p}$ does not fork over $\emptyset$ and $\tilde{p}|B$ is stationary) iff $\mathrm{Cb}(p) \subseteq \operatorname{dcl}(B)$.

Problem 3. (i) Let X be an A-invariant set. Prove that X jest 1-based iff for every [finite] $a \subseteq X$ and every B,

$$a \underset{\operatorname{acl}(Aa) \cap \operatorname{acl}(AB)}{\perp} B.$$

(ii) Conclude that T jest 1-based iff for every A and B, $A \perp_{\operatorname{acl}(A) \cap \operatorname{acl}(B)} B$.
(ii) Prove that 1-basedness of X does not depend on the choice of A over which X is invariant.
Hint. Use the general fact: $a \perp_C b \Rightarrow \operatorname{acl}(a) \cap \operatorname{acl}(b) \subseteq \operatorname{acl}(C)$.

Problem 4*. Prove that condition (ii) in the corollary recalled in Problem 8 can be weakened to (ii') saying that any $\emptyset$-definable subsets of G^n has the required property. Namely, prove that (i) is equivalent to (ii').