Stable groups, List 10

Problem 1. Let K be an algebraically closed field. Prove that if the matrices from a set $G \subseteq GL_n(K)$ commute, then they have a common eigenvector.

Problem 2. Let K be an algebraically closed field. Prove that each solvable subgroup G of $GL_n(K)$ has a finite index subgroup H for which there is $A \in GL_n(K)$ such that for any $B \in H$ the matrix ABA^{-1} is upper triangular.

Problem 3. Prove that if G is a connected, solvable group of finite Morley rank, then G' is nilpotent.
Comment. During the lecture, Michał observed that for definable, connected, solvable groups of matrices over an algebraically closed field this follows from Lie-Kolchin-Malcev theorem.

Problem 4. Let V be a vector space of finite dimension over a field K. Prove that $g \in \mathrm{Aut}(V)$ is unipotent if and only if in some basis the matrix of g is upper triangular with 1's on the diagonal.

Problem 5. Assume A is an abelian group, $\{0\} = B_0 < B_1 \cdots < B_n = A$, $G \leq \mathrm{Aut}(A)$, and for every i the group B_i is G-invariant and G acts on B_i/B_{i-1} trivially. Show that G is nilpotent of class $\leq n$.

Problem 6. Let p be a prime number. Prove that each connected p-group (in the weak sense) of finite Morley rank is either nilpotent or has a bad subgroup, where we say that a group G is a *p-group in the weak sense* if for every definable $H_2 \lhd H_1 \leq G$ with H_1/H_2 infinite there is an element of order p in H_1/H_2.
Hint. First notice that the problem reduces (and is even equivalent) to showing that a connected, solvable p-group (in the weak sense) of finite Morley rank is nilpotent. Then follow the lines of the proof of the theorem that each connected, solvable but not nilpotent group of finite Morley rank interprets an infinite field (it is important where exactly this field lives).

Problem 7. Prove that a connected group of finite Morely rank and of finite exponent is either nilpotent or has a bad subgroup.

Problem 8. Prove that an infinite group of finite Morely rank has infinitely many conjugacy classes or has a bad subgroup.