Stable groups, List 12

(G, S) is a type-definable (in a monster model $\mathfrak{C}$ of a theory T) homogeneous space.

Problem 1. Prove that:
(i) the group operation on G is a relatively definable subset of G^3; even more: there is a formula $\theta(x, y, z)$ such that $\theta(G)$ is the group operation and $\forall x, y \exists! z \theta(x, y, z)$ (for x, y, z realizing $\theta(x, y, z)$ we write $x \cdot y = z$),
(ii) the action of G on S is a relatively definable subset of $G \times S \times S$; even more: there is a formula $\zeta(x, y, z)$ such that $\zeta(\mathfrak{C}) \cap (G \times S \times S)$ is the action of G on S and $\forall x \forall y \exists! z \zeta(x, y, z)$ (for x, y, z realizing $\zeta(x, y, z)$ we write $x * y = z$),
(iii) the function of taking the inverse of an element of G is a relatively definable subset of $G \times G$; even more: there is a formula $i(x, y)$ such that $i(G)$ is the graph of inversion and $\forall x \exists! y i(x, y)$ (for x, y realizing $i(x, y)$ we write $y = x^{-1}$).

Problem 2. Let $\Phi(x)$ and $\Psi(x)$ be partial types closed under conjunction and defining G and S, respectively. Consider formulas θ, ζ, i, from the previous problem. Prove that there exist formulas $\varphi_0(x) \in \Phi(x)$ and $\psi_0(x) \in \Psi(x)$ such that:
(i) $(\forall x, y, z \in \varphi_0(\mathfrak{C}))((x \cdot y) \cdot z = x \cdot (y \cdot z))$,
(ii)$(\forall x \in \varphi_0(\mathfrak{C}))(\forall y_1, y_2 \in \varphi_0(\mathfrak{C}))(y_1 \neq y_2 \rightarrow (x \cdot y_1 \neq x \cdot y_2 \wedge y_1 \cdot x \neq y_2 \cdot x))$,
(iii) $(\forall x \in \varphi_0(\mathfrak{C}))(e \cdot x = x \cdot e = x)$,
(iv) $(\forall x \in \varphi_0(\mathfrak{C}))(x \cdot x^{-1} = x^{-1} \cdot x = e)$.
(v) $(\forall x_1, x_2 \in \varphi_0(\mathfrak{C}))(\forall y \in \psi_0(\mathfrak{C}))(x_1 * (x_2 * y) = (x_1 \cdot x_2) * y)$,
(vi) $(\forall x \in \varphi_0(\mathfrak{C}))(\forall y_1, y_2 \in \psi_0(\mathfrak{C}))(y_1 \neq y_2 \rightarrow x * y_1 \neq x * y_2)$,
(vii) $(\forall y \in \psi_0(\mathfrak{C}))(e * y = y)$.
Notice that for $\varphi_0(x)$ and $\psi_0(x)$ with the above properties we additionally have:
(i') $(\forall g, g_1, g_2 \in G)(\forall x \in \varphi_0(\mathfrak{C}))(g_1 \cdot (g_2 \cdot (g \cdot x)) = (g_1 \cdot g_2) \cdot (g \cdot x))$,
(ii') $(\forall g \in G)(\forall y_1, y_2 \in G \cdot \varphi_0(\mathfrak{C}))(y_1 \neq y_2 \rightarrow g \cdot y_1 \neq g \cdot y_2)$,
(iii') $(\forall g, g_1, g_2 \in G)(\forall x \in \psi_0(\mathfrak{C}))(g_1 * (g_2 * (g * x)) = (g_1 \cdot g_2) * (g * x))$,
(iv') $(\forall g \in G)(\forall y_1, y_2 \in G * \psi_0(\mathfrak{C}))(y_1 \neq y_2 \rightarrow g * y_1 \neq g * y_2)$.

Problem 3. Let $X \subseteq S$ be relatively definable and $g \in G$. Prove that $g * X$ and X^{-1} are both relatively definable in G.

Problem 4. Assume that T is stable. Let $X \subseteq S$ be relatively definable. We define a 2-sorted structure $C_0 := (S, G; R)$, where $C_0 \models R(x, y)$ iff $x \in S$, $y \in G$ and $x \in y * X$.
(i) Prove that $R(x, y)$ is stable in $\mathrm{Th}(C_0)$.
(ii) During the lecture, it was shown that X is generic iff $R(x, e)$ does not divide over $\emptyset$ in the sense of $\mathrm{Th}(C_0)$. Note that the same proof yields that $S \setminus X$ is generic iff $\neg R(x, 1)$ does not divide over $\emptyset$ in $\mathrm{Th}(C_0)$.

Problem 5. Assume that T is stable. Prove that there exists a generic type $p(x) \in S(\mathfrak{C}) \cap [\Psi(x)] = S_S(\mathfrak{C})$ (where $\Psi(\mathfrak{C}) = S$).

Problem 6. Assume T is stable. Prove directly (pointing out the order property if the conclusion fails) that if $X \subseteq G$ is relatively definable, then X is a 2-sided generic or $G \setminus X$ is a 2-sided generic.

Problem 7. (i) Let Δ be a finite set of formulas and let $\pi(x)$ be a type over some small $A \subset \mathfrak{C}$. Prove that $R_\Delta(\pi(x)) = CB([\pi(x)]_\Delta)$, where $[\pi(x)]_\Delta := \{p(x) \in S_\Delta(\mathfrak{C}) : p(x) \cup \pi(x) \text{ is consistent}\}$.
(ii) Prove the same for stratified ranks R_{Δ_φ} defined during the lecture.