Stable groups, List 13

(G, S) is a type-definable (in a monster model $\mathfrak{C}$ of a theory T) homogeneous space; $\Phi(x)$ defines G and $\Psi(x)$ defines S.

Problem 1. Let $X \subseteq S$ be relatively Δ_φ-definable and $g \in G$. Prove that $g * X$ is also relatively Δ_φ-definable in S.

Problem 2. Prove that:
(i) G^0 is an intersection of $\leq |T|$ relatively $\emptyset$-definable subgroups of G (and so G^0 is a $\emptyset$-type-definable subgroup of G of index $\leq 2^{|T|}$),
(ii) $G^0 \trianglelefteq G$,
(iii) $(G^0)^0 = G^0$.

Problem 3. Let $f : Gen \to \lim Gen_{\Delta_\varphi}$ be defined as follows:

$$f(p) := (p \restriction \Delta_\varphi : \varphi \in \mathcal{L}).$$

Prove that:
(i) f is a homeomorphism,
(ii) $f(g * p) = g * f(p)$ for $g \in G$ and $p \in Gen$,
(iii) for $p \in Gen$ the orbit $G * p$ is closed and dense in Gen, and so equal to Gen. (This was briefly explained during the lecture.)

Problem 4. Prove that if $p(x) \in S_S(\mathfrak{C})$ is generic, then for every $g \in G$ the type $g * p$ does not fork over $\emptyset$.

Problem 5. Justify that the construction of generics via Δ_φ-ranks presented during the lecture indeed leads generics.

Problem 6. Prove that if $a \in S$ is generic over A and $g \mathop{\smile\!\!\!\!\!\!\mid}_A a$, then $g * a$ is generic in S over A.

Problem 7. Let $p \in S(A)$ and $q \in S(B)$ be types extending $\Phi(x)$, and assume that p is a generic type of G. Prove that there are $g \in G$, such that $(g \cdot p) \cup q$ is a non-forking (so in particular consistent) extension of q.