Stable groups, List 3

Problem 1. Let G be an arbitrary group. Recall that $\mathrm{Socle}(G)$ is the subgroup generated by all the minimal normal subgroups of G.
(i) Prove that if $H \neq K$ are minimal normal subgroups of G, then $H \cap K = \{e\}$ and H and K commute.
(ii) Prove that if G is finite and H is a minimal normal subgroup of G, then $\mathrm{Socle}(H) = H$.
(iii) Prove that if G is finite and H is a minimal normal subgroup of G, then $H = K_1 \times \cdots \times K_n$ for some normal subgroups $K_1, \ldots, K_n$ of H, which are all simple and commute with each other.

Problem 2. Let G be an ω-categorical, stable group. Prove that the minimal normal subgroups of G are uniformly definable, and that at least one such a subgroup exists.

Problem 3. We work in an ω-categorical structure. Let M be a definable, abelian group acting definably and by automorphisms on a definable abelian group A. Prove that the family $\{\langle Ma \rangle : a \in A\}$ is uniformely definable.

Problem 4. In the context from the previous problem, let $B := \langle Ma \rangle$ be infinite. Let R be the ring of endomprhisms of B generated by M. Prove that R is interpretable (in the structure in which we are working).

Problem 5. Let G be a connected group definable in an ω-categorical, stable structure. Prove that G' is also connected.

Problem 6. (i) Prove that each virtually abelian group has a definable, abelian subgroup of finite index.
(ii) Prove that each virtually nilpotent group has a definable, nilpotent subgroup of finite index.
(iii) Prove that each virtually solvable group has a definable, solvable subgroup of finite index.

Problem 7. Prove that an infinite Boolean algebra is unstable.

Problem 8. Prove that a formally real field is unstable.

Problem 9. Prove that a group with a single non-trivial conjugacy class is unstable.

Problem 10. (i) Prove that in a stable group, there is a finite bound on the number of pairwise commuting, non-abelain, normal subgroups.
(ii) Prove that in a stable group, among minimal, normal subgroups there are only finitely many non-abelian ones.