Topological dynamics in model theory. List 1.

Problem 1. Consider the flow $(\mathbb{Z}, S^1 \times S^1)$ with the action defined by $n * (x, y) := (\alpha^n x, \beta^n y)$, where $\alpha, \beta \in S^1$. Prove that $(\mathbb{Z}, S^1 \times S^1)$ is minimal if and only if $(\forall (m,n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\})(\alpha^n \beta^m \neq 1)$.

Problem 2. Construct an almost periodic element in the Bernoulli shift $(\mathbb{Z}, 2^{\mathbb{Z}})$ which generates an infinite (minimal) subflow.

Problem 3. Let G be a topological group and (G, X) a G-flow. Prove that if $p \in X$ is almost periodic, then for every open $U \ni p$ the set $\{g \in G : gp \in U\}$ is (left) generic (and so (right) syndetic).

Problem 4. Let (G, X) be any flow and $p \in X$. Prove that the following conditions are equivalent.

(i) p is almost periodic.

(ii) For every open $U \ni p$ there exists a finite $A \subseteq G$ with $\mathrm{cl}(Gp) \subseteq AU$.

(iii) For every open subset U of X such that $U \cap \mathrm{cl}(Gp) \neq \emptyset$ there exists a finite $A \subseteq G$ with $\mathrm{cl}(Gp) \subseteq AU$.

Problem 5. Let (G, X) be any G-set and $D \subseteq X$. Prove that the following conditions are equivalent.

(i) D is weakly generic.

(ii) There is a finite $F \subseteq G$ such that $X \setminus FD$ is not generic.

(iii) D is piecewise syndetic.

Problem 6. Let M be any structure and $N \succ M$ an $|M|^+$-saturated elementary extension. Let G be a group 0-definable in M.

(i) Prove that the definition of $S_{G,ext}(M)$ does not depend on the choice of N.

(ii) Prove that the G-flows $S_{G,ext}(M)$ and $S_{G,M}(N)$ are isomorphic.

Problem 7. Let G be a 0-definable group in a structure M, and $D \subseteq G$ definable. Prove that the following conditions are equivalent.

(i) D is weak generic.

(ii) There is a definable $V \subseteq G$ which is not generic but $D \cup V$ is generic.

Problem 8. Let G be a 0-definable group in a structure M, and $D \subseteq G$ definable. Let $N \succ M$ be an elementary extension. Prove that:

(i) if D is weak generic, then $D(N)$ is also weak generic (as a subset of $G(N)$),

(ii) the converse holds provided that M is $\aleph_0$-saturated.

Problem 9. Let T be a stable theory, $p(x) \in S(\emptyset)$, and $\varphi(x)$ a formula with parameters. Prove that:

(i) $\varphi(x)$ is c-free over p if and only if $p(x) \cup \{\varphi(x)\}$ does not fork over $\emptyset$,

(ii) $\varphi(x)$ is c-free over some $q \in S(\emptyset)$ if and only if $\varphi(x)$ does not fork over $\emptyset$.