Topological dynamics in model theory. List 10.

Let $\mathfrak{C}$ be a monster model of a complete theory T.

Problem 1.

(i) Prove that every set $A \subseteq \mathfrak{C}$ is contained in an ambitious model of cardinality at most $|A| + |T| + \aleph_0$. In particular, every countable theory has a countable ambitious model.

(ii) Prove that if G is a group which is $\emptyset$-type-definable in $\mathfrak{C}$, then there is a countable G-ambitious model.

Problem 2. Prove that if a small model $M \prec \mathfrak{C}$ is ambitious and $\bar{m}$ is its enumeration, then the function $\hat{f} \colon E(S_{\bar{m}}(M)) \to \mathrm{Gal}_L(T)$ given by $\hat{f}(\eta) := \sigma / \mathrm{Autf}_L(\mathfrak{C})$, where $\sigma \in \mathrm{Aut}(\mathfrak{C})$ is any automorphism such that $\eta(\mathrm{tp}(\bar{m}/M)) = \mathrm{tp}(\sigma(\bar{m})/M)$, is onto.

Problem 3. Prove the descriptive set theoretic lemma from page 64.

Problem 4. Let $p(\bar{x}) \in S(\emptyset)$, $\bar{\alpha} \models p$, and let E be a strong type on $p(\mathfrak{C})$ (i.e. a bounded, invariant equivalence relation on $p(\mathfrak{C})$) coarser than E_{KP}. Let $r_{[\bar{\alpha}]_E} \colon \mathrm{Gal}_{KP}(T) \to p(\mathfrak{C})/E$ be given by $r_{[\bar{\alpha}]_E}(\sigma / \mathrm{Autf}_{KP}(\mathfrak{C})) := [\sigma(\bar{\alpha})]_E$. Prove that $r_{[\bar{\alpha}]_E}$ is a topological quotient map

Problem 5. Let (G, X, x_0) be any G-ambit. Let $\pi_0 \colon E(X) \to X$ be given by $\pi_0(\eta) := \eta(x_0)$.

(i) Prove that for every $\eta_1, \eta_2 \in E(X)$ we have $\pi_0(\eta_1 \eta_2) = \eta_1(\pi_0(\eta_2))$.

(ii) Let $\equiv$ be the fiber equivalence relation of π_0 on $E(X)$. Let $\mathcal{M}$ be a minimal left ideal of $E(X)$ and u an idempotent in $\mathcal{M}$. Let $D := [u]_\equiv \cap u\mathcal{M}$. Prove that D is a τ-closed subgroup of $u\mathcal{M}$.