Topological dynamics in model theory. List 11.

Let $\mathfrak{C}$ be a monster model of a complete theory T. Let $\mathfrak{C}' \succ \mathfrak{C}$ be a monster model with respect to $|\mathfrak{C}|$.

Problem 1. Assume T is stable. Prove that the "restriction" map $R \colon NF_{\bar{c}}(\mathfrak{C}) \to \mathrm{Aut}(\mathrm{acl}^{eq}(\emptyset))$ given by $R(\mathrm{tp}(\sigma'(\bar{c})/\mathfrak{C})) := \sigma' \restriction_{\mathrm{acl}^{eq}(\emptyset)}$ (where $\sigma' \in \mathrm{Aut}(\mathfrak{C}')$) is an isomorphism of semigroups and of $\mathrm{Aut}(\mathfrak{C})$-flows. Deduce that the semigroup $NF_{\bar{c}}(\mathfrak{C})$ is a topological group.

Problem 2. Let $p, q, r \in NF_{\bar{c}}(\mathfrak{C})$. Take any $\bar{c}' \models q$. Prove that $p * q = r$ if and only if there exists $\sigma' \in \mathrm{Aut}(\mathfrak{C}')$ such that $\sigma'(\bar{c}) \models p$ and $\sigma'(\bar{c}') \models r$.

Problem 3. Let T be the theory of the circular order $(S^1, C(x, y, z))$. Prove that it has quantifier elimination and is ω-categorical with the unique countable model $(\mathbb{Q}/\mathbb{Z}, C(x, y, z))$. Deduce that $|S_1(\emptyset)| = 1$.

Problem 4. Let T be the theory of the structure $M_n := (\mathbb{Q}/\mathbb{Z}, R_n(x), C(x, y, z))$, where $C(x, y, z)$ is the (clockwise) circular order, and $R_n(x/\mathbb{Z}) := (x + \frac{1}{n})/\mathbb{Z}$. Prove that it has quantifier elimination and is ω-categorical, and $(S^1, R_n, C_n(x, y, z)) \succ M_n$, where R_n and C_n are defined as in M_n. Deduce that $|S_1(\emptyset)| = 1$.

Problem 5. Prove the proposition on page 72.

Problem 6. Let (G, X) be any flow. Let $\mathrm{Inv}_G(X) := \{x \in X : Gx = \{x\}\}$.

(i) Prove that for every $\eta \in E(X)$ and $x \in \mathrm{Inv}_G(X)$, $\eta(x) = x$. Deduce that $\mathrm{Inv}_G(X) \subseteq \mathrm{Im}(\eta)$.

(ii) Prove that if there is $\eta \in E(X)$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_G(X)$, then the Ellis group of X is trivial.

Problem 7. Let $\mathbb{K}$ be the random graph which is partitioned into finitely many substructures $B_0, \ldots, B_{r-1}$. Prove that $B_i \cong \mathbb{K}$ for some $i < r$.