Topological dynamics in model theory. List 14.

Let M be a model in a language L, and $\varphi(x,y)$ an L-formula.

Problem 1. Let X be a compact space, and let $A \subseteq C(X)$ be a (norm) bounded set of functions. Prove that A is pointwise precompact if and only if the pointwise closure of A is contained in $C(X)$.

Problem 2. Prove that the following conditions are equivalent.

(i) $\varphi(x,y)$ is stable in M, i.e. there do NOT exist sequences $(a_i)_{i\in\omega}$ and $(b_j)_{j\in\omega}$ of elements of M such that: $(\forall i,j)(M \models \varphi(a_i,b_j) \iff i \leq j)$ or $(\forall i,j)(M \models \neg\varphi(a_i,b_j) \iff i \leq j)$. (So the definition from the lecture should be corrected by adding the second possibility.)

(ii) There are no sequences $(a_i)_{i\in\omega}$ and $(b_j)_{j\in\omega}$ in M such that: $(\forall_i^\infty \forall_j^\infty \varphi(a_i,b_j)$ and $\forall_j^\infty \forall_i^\infty \neg\varphi(a_i,b_j))$ or $((\forall_i^\infty \forall_j^\infty \neg\varphi(a_i,b_j)$ and $\forall_j^\infty \forall_i^\infty \varphi(a_i,b_j))$.

(iii) For every sequences $(a_i)_{i\in\omega}$ and $(b_j)_{j\in\omega}$ in M, $\lim_i \lim_j \varphi(a_i,b_j) = \lim_j \lim_i \varphi(a_i,b_j)$ whenever both limits exist.

Problem 3. Let $\mathfrak{C}' \succ \mathfrak{C} \succ M$ be monster models of $\mathrm{Th}(M)$ such that $\mathfrak{C}'$ is a monster with respect to $\mathfrak{C}$. Let A be the set of all functions from $S_y(M)$ to $\{0,1\}$ of the form $\chi_{\tilde{\varphi}(y,a)}$ where $a \in M$. Let $f\colon S_y(M) \to \{0,1\}$. Prove that f belongs to the pointwise closure of A if and only if there is $a' \in \mathfrak{C}'$ such that $\mathrm{tp}(a'/\mathfrak{C})$ is finitely satisfiable in M and for every $q \in S_y(M)$, $f(q) = \varphi(a',b)$ for some/any $b \in q(\mathfrak{C})$.

Problem 4. Let (G,X) be a flow. Show that the WAP functions in $C(X)$ form a closed subalgebra.