Topological dynamics in model theory. List 2.

$\mathfrak{C}$ always denotes a monster model of a complete theory.

Problem 1.

(i) For a group G definable in a model M prove that $(G, S_G(M))$ is a flow (with the obvious action of G on $S_G(M)$ defined in Lecture 1).

(ii) For a partial type $\pi(x)$ over $\emptyset$ prove that $(\mathrm{Aut}(\mathfrak{C}), S_\pi(\mathfrak{C}))$ is a flow, where $\mathrm{Aut}(\mathfrak{C})$ is equipped with the pointwise convergence topology.

Problem 2. Prove that the relation $R(x, y)$ on $\mathfrak{C}$ saying that there exists a model $M \prec \mathfrak{C}$ such that $x \equiv_M y$ is 0-type-definable.

Problem 3. Let $p \in S(\emptyset)$ and let E be a bounded, invariant equivalence relation on $p(\mathfrak{C})$. Prove that E is type-definable if and only if it has a type-definable class.

Problem 4. Let G be a 0-definable group in a model M, and let $N = (M, \cdot, X)$ be the structure obtained from M by adding an affine copy X of G as a new sort (as explained in the lecture). Prove that $\mathrm{Aut}(N) = G \rtimes \mathrm{Aut}(M)$.

Problem 5. Let X be a definable set in a model M, and let C be a compact (Hausdorff) space. Prove the following statements.

(i) If $f \colon X \to C$ is definable, then it extends uniquely to an M-definable function $f^* \colon X(\mathfrak{C}) \to C$. Moreover, f^* is given by $\{f^*(a)\} = \bigcap_{\varphi \in \mathrm{tp}(a/M)} \mathrm{cl}(f[\varphi(M)])$.

(ii) Conversely, if $f^* \colon X(\mathfrak{C}) \to C$ is an M-definable function, then $f^*|_X \colon X \to C$ is definable.

(iii) A function $f^* \colon X(\mathfrak{C}) \to C$ is definable over M if and only if there is a continuous function $h \colon S_X(M) \to C$ such that $f^* = h \circ r$, where $r \colon X(\mathfrak{C}) \to S_X(M)$ is given by $r(a) := \mathrm{tp}(a/M)$.

Problem 6. Let X be a definable set in a model M. Prove that the map $r \colon X \to S_X(M)$ given by $r(a) := \mathrm{tp}(a/M)$ is a definable compactification of X.

Problem 7. Let X and r be as in Problem 6. Prove that $r^* \colon X(\mathfrak{C}) \to S_X(M)$ given by $r^*(a) := \mathrm{tp}(a/M)$ is the unique extension of r to an M-definable function from $X(\mathfrak{C})$ to $S_X(M)$ (see Problem 5 for uniqueness).

Problem 8. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a Boolean algebra of subsets of a set X. Let μ be a finitely additive measure on $\mathcal{A}$. Consider any family $\{A_n\}_{n<\omega}$ of sets from $\mathcal{A}$ and $\epsilon > 0$ such that $\mu(A_n) > \epsilon$ for all $n < \omega$. Prove that there exists an increasing sequence $(n_k)_{k<\omega}$ of natural numbers such that $\mu(A_{n_0} \cap \cdots \cap A_{n_k}) > 0$ for all $k < \omega$.