Topological dynamics in model theory. List 3.

Problem 1. Check that for every set X, $(X^X, \circ)$ is a left topological semigroup.

Problem 2. Prove that if S is a semigroup with left identity and left inverses (with respect to some left identity), then S is a group.

Problem 3. Let S be left topological semigroup, $\mathcal{M}$, $\mathcal{N}$ minimal left ideals in S, and $u \in \mathcal{M}$, $v \in \mathcal{N}$ idempotents such that $uv = v$ and $vu = u$. Prove that the assignment $x \mapsto xv$ defines a group isomorphism from $u\mathcal{M}$ to $v\mathcal{N}$.

Problem 4. Let (G, X) be a G-flow. Show that $\mathcal{M} \subseteq E(X)$ is a minimal left ideal if and only if it is a minimal subflow.

Problem 5. Let (G, X) be a G-flow. Show that any two minimal subflows of $E(X)$ are isomoprhic (as flows, not as semigroups).

Problem 6. Let X be any set, and let $H \subseteq X^X$ be a group (with composition as group operation). Prove that:

(i) all $h \in H$ have the same image, which we denote by I,

(ii) $F \colon H \to \mathrm{Sym}(I)$ given by $F(h) := h|_I$ is a group monomorphism.

Problem 7. Let (G, X) be a G-flow, and $\mathcal{M}$ a minimal left ideal. Show that for every $\eta \in E(X)$ there is an idempotent $u \in \mathcal{M}$ such that $\mathrm{Im}(u) \subseteq \mathrm{Im}(\eta)$.

Problem 8. Let $(G, \mathcal{U}, u_0)$ be the universal G-ambit in some category $\mathcal{C}$ of G-flows which closed under taking subflows. For any $u \in \mathcal{U}$, let $f_u \colon \mathcal{U} \to \mathcal{U}$ be the unique homomorphism of flows mapping u_0 to u. Define $*$ on $\mathcal{U}$ by $p * u := f_u(p)$. Prove that $*$ is associative. (This is a part of the fact from the lecture that $*$ is a left continuous semigroup operation on $\mathcal{U}$ extending the action of G.)

Define also a natural left continuous action of the semigroup $(\mathcal{U}, *)$ on an arbitrary flow from $\mathcal{C}$.

Problem 9.

(i) Observe that if the Ellis semigroup $E(X)$ of a flow (G, X) is a group, then it is a unique minimal left ideal of $E(X)$ and it is also the Ellis group of (G, X).

(ii) Consider the flow $(\mathbb{Z}, S^1)$ with the action $n \cdot z := \alpha^n \cdot z$, where $\alpha \in S^1$ is not a root of unity. Prove that $E(S^1)$ is topologically isomorphic to the group S^1. Conclude that it coincides with the Ellis group of $(\mathbb{Z}, S^1)$.

(iii) Let $X := S^1 \times S^1$, $G := \mathbb{Z}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, and $\psi \colon S^1 \to S^1$ be continuous. Consider the flow (G, X) given by $1 \cdot (x_1, x_2) := (e^{2\pi\alpha i}x_1, \psi(x_1)x_2)$. Prove that $E(X)$ is a group which consists of all functions $f \colon X \to X$ of the form $f(x_1, x_2) := (e^{2\pi\beta i}x_1, \varphi(x_1)x_2)$, where $\beta \in \mathbb{R}$, $\varphi \colon S^1 \to S^1$, and $\psi(e^{2\pi\alpha i}x)/\psi(x) = \varphi(e^{2\pi\beta i}x)/\varphi(x)$ for all x.

Comment. It is possible that (iii) is wrong. If it is so, correct it.

Problem 10. Consider the Bernoulli shift $(\mathbb{Z}, 2^{\mathbb{Z}})$. Show that that minimal left ideals of $E(2^{\mathbb{Z}})$ are proper subsets of $E(2^{\mathbb{Z}})$. Conclude that $E(2^{\mathbb{Z}})$ is not a group.