# Topological dynamics in model theory. List 5.

$\mathfrak{C}$ always denotes a monster model of a complete theory. For a group $G$ definable in $M \prec \mathfrak{C}$, $G^* := G(\mathfrak{C})$.

**Problem 1.** Let $G$ be a group $\emptyset$-type-definable in a monster model, and $H \lhd G$ be an invariant subgroup of bounded index. Prove that $G/H$ equipped with the logic topology is a topological group (i.e. the group operation and inversion are continuous).
*Hint. First, prove it for $H$ type-definable, and then use it in general.*

**Problem 2.** Prove that (where $\cong$ denotes *topological* isomorphisms):

(i) for any $\emptyset$-type-definable (in $\mathfrak{C}$) group $G$ , $G/G^0_A$ is always profinite;

(ii) for $M := (\mathbb{Z}, +)$ and $G(M) := \mathbb{Z}$, $G^*/G^{*00} = G^*/G^{*0} \cong \hat{Z}$;

(iii) for $M := (\mathbb{R}, +, \cdot)$ and $G(M) := S^1$, $G^*/G^{*00} \cong S^1$ and $G^*/G^{*0}$ is trivial;

(iv) more generally: Let $G = (G, \cdot, \dots)$ be a compact topological group with a basis of neighborhoods of the identity consisting of definable subsets. Check that the standard part map $\mathrm{st}\colon G^* \to G$ is a well-defined homomorphism. Assume that all definable subsets of $G$ have the Baire Property. Prove that $\ker(\mathrm{st}) = G^{*00}_G$ and $G/G^{*00}_G \cong G$.

**Problem 3.** Let $G$ be a group definable in $M$. Prove that the quotient map $G \to G^*/G^{*00}_M$ is a definable compactification of $G$.

**Problem 4.** Prove that the map $\hat{f}\colon S_{G,M}(N) \to G^*/G^{*000}_A$ given by $\hat{f}(\mathrm{tp}(a/N)) = a/G^{*000}_A$ is continuous (where $A \subseteq M$).

**Problem 5.** Prove that every externally definable subset of $\mathbb{R}$ (in $M := (\mathbb{R}, +, \cdot)$) is definable. Deduce that every externally definable subset of $S^1$ is definable.

**Problem 6.** Let $M := (\mathbb{R}, +, \cdot)$ and $G := S^1$.

(i) Prove that $p_1^-$ and $p_1^+$ are the only idempotents in $Gen$.

(ii) Prove that for every $\epsilon \in \{-, +\}$, $p_1^\epsilon * Gen = \{p_a^\epsilon : a \in G\}$.

**Problem 7.** Let $K$ be a compact group definable in an o-minimal expansion of the reals. For an idempotent $u \in S_K(\mathbb{R})$ and $k \in K$, by $u(k)$ we denote the unique type in $u * Gen$ with $\mathrm{st}(u(k)) = k$. Prove that for every idempotents $u, u' \in S_K(\mathbb{R})$ and elements $k, k' \in K$, $u(k) * u'(k') = u(kk')$.

**Problem 8.** Choose elements $b, c$ in an elementary extension of $(R, \cdot, +)$ so that $b > \mathbb{R}$ and $c > \mathbb{R}(b)$. Let $p_0$ be the type of the matrix $\begin{bmatrix} b & c \\ 0 & b^{-1} \end{bmatrix}$ over $\mathbb{R}$. Prove that $H \cdot p_0 = \{p_0\}$, where $H$ is the group of upper-triangular $2 \times 2$-matrices of determinant 1 with positive elements on the diagonal.

**Problem 9** Let $G$ be a group definable in $M$ such that $G = KH$ for some definable subgroups $K$ and $H$ with $H \cap K = \{e\}$. Then $H$ acts naturally on $K$ by $\varphi_h(k) = h \cdot k := k'$ for a unique $k' \in K$ such that $hk = k'h'$ for some (unique) $h' \in H$. And the same holds in the monster model.

Assume that all types in $S_G(M)$ are definable. Prove that the semigroup $(S_H(M), *)$ acts on $S_K(M)$ by $p \cdot q := \mathrm{tp}(\varphi_h(k)/M)$ for every [some] $k \models p$ and $h$ satisfying the unique heir extension of $q$ over $M, k$.

*Hint. Prove that if all types in $S_G(M)$ are definable, then for every $p, q, r \in S_G(M)$, if $a \models p$, $b \models q|_{M,a}$, and $c \models r|_{M,a,b}$ (i.e. heir extensions), then $\mathrm{tp}(a/M, b, c)$ is a coheir extension of $p$.*