Topological dynamics in model theory. List 7.

In Problems 3-7, (G, X) is a flow, $\mathcal{M}$ a minimal left ideal in $E(X)$, and $u \in J(\mathcal{M})$.

Problem 1. Prove that for a semitopological group G and $A \subseteq G$, we have $\bar{A} = \bigcap\{V^{-1}A : V \text{ an open neighborhood of } e\}$.

Problem 2. Let $f\colon X \to Y$ be a continuous map from a topological space X to a Hausdorff space Y. Prove that for every $x \in X$, f is constant on the set $\bigcap\{\bar{V} : V \text{ an open neighborhood of } x\}$.

Problem 3. Let $B, C \subseteq E(X)$, and $a, b \in E(X)$. Prove that:

(i) $(a \circ B)c = a \circ (Bc)$,

(ii) $a \circ (b \circ B) \subseteq (ab) \circ B$,

(iii) $aB \subseteq a \circ B$,

(iv) $a \circ (B \cup C) = (a \circ B) \cup (a \circ C)$,

(v) $a \circ (bC) \subseteq (ab) \circ C$ and $a(b \circ C) \subseteq (ab) \circ C$.

Problem 4. Let $a \in E(X)$, I be a closed left ideal in $E(X)$, and $B \subseteq I$. Prove that $a \circ B \subseteq I$.

Problem 5. Prove that if a net $(a_i)_i$ in $u\mathcal{M}$ converges to $a \in \overline{u\mathcal{M}}$ (in the usual topology on $E(X)$), then $(a_i)_i$ converges to ua in the τ-topology on $u\mathcal{M}$.

Problem 6.

(i) Prove that for every $A \subseteq u\mathcal{M}$ and $\eta \in \mathcal{M}$, $\eta \circ A$ is the collection of all $b \in E(X)$ for which there exist nets $(\eta_i)_i$ in $\mathcal{M}$ and $(a_i)_i$ in A such that $\lim \eta_i = \eta$ and $\lim \eta_i a_i = b$.

(ii) Conclude that for every $A \subseteq u\mathcal{M}$, $\mathrm{cl}_\tau(A)$ is the collection of all $b \in u\mathcal{M}$ for which there exist nets $(\eta_i)_i$ in $\mathcal{M}$ and $(a_i)_i$ in A such that $\lim \eta_i = u$ and $\lim \eta_i a_i = b$.

Problem 7. For $p \in u\mathcal{M}$, put $\Gamma_p := \mathrm{graph}(r_p) := \{(x, xp) : x \in \mathcal{M}\}$. For $A \subseteq u\mathcal{M}$, put $\Gamma_A := \bigcup_{p \in A} \Gamma_p$. Prove that for every $A \subseteq u\mathcal{M}$, $\mathrm{cl}_\tau(A) = \{p \in u\mathcal{M} : \Gamma_p \subseteq \overline{\Gamma_A}\}$.