Topological dynamics in model theory. List 8.

Let G be a group definable in M, $A \subseteq M$, $N \succ M$ an $|M|^+$-saturated model, $\mathcal{M} \lhd S_{G,M}(N)$ a minimal left ideal, $u \in J(\mathcal{M})$, $\hat{f} \colon S_{G,M}(N) \to G^*/G^{*000}_A$ the epimorphism given by $\hat{f}(\mathrm{tp}(a/N)) := a/G^{*000}_A$, $f := \hat{f}\restriction_{u\mathcal{M}}$, and $\theta := \rho \circ f \colon u\mathcal{M} \to G^*/G^{*00}_A$ (where $\rho \colon G^*/G^{*000}_A \to G^*/G^{*00}_A$ is the obvious map). Let also $P_u := \ker(f)$ and $S := \mathrm{cl}_\tau(P_u) = u(u \circ P_u)$.

Problem 1. Prove that:

(i) $\hat{f}$ is a topological quotient map,

(ii) $\hat{f}\restriction_{\mathcal{M}}$ is a topological quotient map.

Problem 2. Deduce from Theorem 2 on p. 49 that the epimorphism θ is a topological quotient map which factors through the quotient map $\pi : u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$, and that the induced epimorphism $\theta \colon u\mathcal{M}/H(u\mathcal{M}) \to G^*/G^{*00}_A$ is a topological quotient map.

Problem 3. Let $F(x)$ be the type over M saying that $x = yz^{-1}$ for some $x \equiv_M y$. Prove that for every $c \models u$ we have that $\models F(c)$. Deduce that for every such c, $c = a_1 b_1^{-1} a_2 b_2^{-1}$, where each of the 2-element sequences (a_1, b_1) and (a_2, b_2) starts an infinite A-indiscernible sequence.

Problem 4. For $v \in J(\mathcal{M})$ put $P_v = \ker(f_v)$, where $f_v := \hat{f}\restriction_{v\mathcal{M}}$. Prove that for $v, w \in J(\mathcal{M})$ we have $vP_w = P_v$.

Problem 5. Prove that $S = SP_u$.

Problem 6. Prove that $\hat{f}^{-1}[f[S]] \cap \mathcal{M} = J(\mathcal{M})S$.
Hint. Use the fact that $J(\mathcal{M}) \subseteq \ker(\hat{f})$ and Ellis theorem.

Problem 7. Here, let (G, X) be an arbitrary flow. Prove that the relation $P(x, y)$ on X saying that x and y are proximal is an equivalence relation if and only if $E(X)$ contains a unique minimal left ideal.
Hint. To prove $(\to)$, consider any two minimal left ideals $\mathcal{M}$ and $\mathcal{N}$ of $E(X)$ and idempotents $u \in J(\mathcal{M})$ and $v \in J(\mathcal{N})$ satisfying $uv = v$ and $vu = u$.

Problem 8. Let $X := S^1$, $t \colon X \to X$ be given by $t(e^{2\pi i\theta}) := e^{2\pi i\theta^2}$, and $s \colon X \to X$ by $s(e^{2\pi i\theta}) = e^{2\pi i(\theta+\beta)}$ for some irrational $\beta \in [0, 1)$ (where $\theta \in [0, 1)$). Let G be the group generated by s, t in the group of homeomorphisms of X. Prove that the nontrivial, minimal G-flow X is proximal. (This implies that G is not strongly amenable.)