Topological dynamics in model theory. List 9.

Let $\mathfrak{C}$ be a monster model of a complete theory T and $\bar{c}$ be its enumeration. Let EL be the Ellis semigroup of the $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c}}(\mathfrak{C})$. $\mathcal{M}$ is a minimal left ideal in EL and $u \in J(\mathcal{M})$. Let $\mathfrak{C}' \succ \mathfrak{C}$ be a monster model in which $\mathfrak{C}$ is small.

Problem 1. Let $\bar{a}$ be any tuple in $\mathfrak{C}$ (possibly $\bar{a} = \bar{c}$). Show that the $\mathrm{Aut}(\mathfrak{C})$-orbit of $\mathrm{tp}(\bar{a}/\mathfrak{C})$ in $S_{\bar{a}}(\mathfrak{C})$ is dense. Deduce that the function $\pi_0 \colon EL \to S_{\bar{c}}(\mathfrak{C})$ given by $\pi_0(\eta) := \eta(\mathrm{tp}(\bar{c}/\mathfrak{C}))$ is onto.

Problem 2.

(i) Check that the function $\hat{f} \colon EL \to \mathrm{Gal}_L(T)$ given by $\hat{f}(\eta) := \sigma' / \mathrm{Autf}_L(\mathfrak{C}')$, where $\sigma' \in \mathrm{Aut}(\mathfrak{C}')$ is any automorphism such that $\eta(\mathrm{tp}(\bar{c}/\mathfrak{C})) = \mathrm{tp}(\sigma'(\bar{c})/\mathfrak{C})$, is well-defined.

(ii) For the function $\hat{f}$ from (i), show that $\hat{f}(\eta) := \sigma' / \mathrm{Autf}_L(\mathfrak{C}')$ for any $\sigma' \in \mathrm{Aut}(\mathfrak{C}')$ such that $\eta(\mathrm{tp}(\bar{d}/\mathfrak{C})) = \mathrm{tp}(\sigma'(\bar{d})/\mathfrak{C})$ for some $\bar{d} \equiv \bar{c}$ (where $\bar{d}$ is from $\mathfrak{C}'$).

Problem 3. Prove that the function $\hat{f}$ from the previous exercise is a topological quotient map. Deduce that so is $\hat{f}\restriction_{\mathcal{M}}$.

Problem 4. Let $p \in S(\emptyset)$ and E be a bounded, invariant equivalence relation on $p(\mathfrak{C})$. Let $\bar{\alpha} \in p(\mathfrak{C})$. Show that the maps g_E and $\bar{h}_E$ from the bottom of page 57 are topological quotient maps. Deduce that $\bar{h}_E[\mathrm{cl}_\tau(\ker(\bar{h}_E))] = \mathrm{cl}(\{\bar{\alpha}/E\})$.

Problem 5. Prove that every closed equivalence relation on a Polish space is smooth.
Hint. Using Luzin separation theorem, find a separating, countable family of Borel sets.

Problem 6. Prove that E_0 is not smooth.
Hint. Suupose E_0 is smooth. Then there is a countable familly of Borel sets separating classes of E_0. Prove that each member of this family is either meager or comeager. Deduce that E_0 has a comeager class, which is a contradiction.

Problem 7. Deduce from Harringtom-Kechris-Louveau dichotomy that every Borel equivalence relation whose all classes are G_δ is smooth.

Problem 8. Assume the language is countable. Let A be a countable set of parameters and assume that Z is A-invariant. Let B be a countable superset of A. Prove that Z_B is Borel [resp. F_σ, clopen, analytic] if and only if Z_A is such.

Problem 9. Let E be a bounded, invariant equivalence relation, and Y a type-definable, E-saturated subset of the domain of E. Let $M \models T$. Prove that $E\restriction_Y$ is type-definable if and only if $E^M\restriction_{Y_M}$ is closed.