Superrosy fields and valuations

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Abstract

We prove that every non-trivial valuation on an infinite superrosy field of positive characteristic has divisible value group and algebraically closed residue field. In fact, we prove the following more general result. Let K be a field such that for every finite extension L of K and for every natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite and, if $\operatorname{char}(K) = p > 0$ and $f \colon L \to L$ is given by $f(x) = x^p - x$, the index $[L^+ : f[L]]$ is also finite. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and, if $\operatorname{char}(K) > 0$, it has algebraically closed residue field. In the zero characteristic case, we get some partial results of this kind.

We also notice that minimal fields have the property that every non-trivial valuation has divisible value group and algebraically closed residue field.

0 Introduction

A motivation for our work comes from some open structural questions concerning fields in various model-theoretic contexts.

A fundamental theorem says that each infinite superstable field is algebraically closed [18, 4]. An important generalization of superstable theories is the class of supersimple theories and yet more general class of superrosy theories. Superrosy theories with NIP (the non independence property) also form a generalization of superstable theories which is "orthogonal" to supersimple theories in the sense that each supersimple theory with NIP is superstable. Recall that a field K is bounded if for every natural number $n > 0$, K has only finitely many separable extensions of degree n (up to isomorphism over K); equivalently, the absolute Galois group of K is small, i.e., it has only finitely many closed subgroups of any finite index. It is known from [12] that perfect, bounded, PAC (pseudo algebraically closed) fields are supersimple. A well-known conjecture predicts the converse:

Conjecture 1 *Each infinite supersimple field is perfect, bounded and PAC.*

A complementary conjecture on infinite superrosy fields with NIP was formulated in [6].

Conjecture 2 *Each infinite superrosy field with NIP is either algebraically or real closed.*

Recall that both algebraically closed and real closed fields are superrosy with NIP. After dropping the NIP assumption, one has to extend the list of possibilities in the conclusion of the above conjecture. Namely, since perfect, bounded, PAC fields as well as orderable, bounded, PRC (pseudo real closed) fields are known to be superrosy [20, Appendix A], the following conjecture is strongest possible. (See Section 4 for the definition of PRC fields, which is chosen so that PAC fields are PRC).

Conjecture 3 *Each infinite superrosy field is perfect, bounded and PRC.*

It is known that a PAC field is simple if and only if it is bounded [12, 2, 3]; it is supersimple if and only if it is perfect and bounded. Similarly, a PRC field is superrosy if and only if it is perfect and bounded (see Fact 4.1). Thus, in Conjectures 1 and 3, once we know that the field is PAC [resp. PRC], the rest of the conclusion is automatically satisfied. It is also easy to see that Conjecture 3 implies Conjecture 1, because one can show that orderable PRC fields have strict order property, and so they are not simple (see Remark 4.2). By Fact 4.3, Conjecture 3 also implies Conjecture 2.

We will be often talking about definable valuations. Throughout the paper 'definable' means 'definable with parameters'.

There are interesting questions and conjectures concerning NIP fields (without assuming superrosiness). By [13], infinite NIP fields are closed under Artin-Schreier extensions. A. Hasson and S. Shelah formulated some dichotomies between nice algebraic properties of the field in question and the existence of non-trivial definable valuations. In particular, one can expect that the following is true.

Conjecture 4 *Suppose K is an infinite field with NIP with the property that for every finite extension L of K and for every natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite. Then either there is a non-trivial definable valuation on K, or K is either algebraically or real closed.*

Note that if a pure field K is algebraically or real closed, then there is no non-trivial definable valuation on K (e.g. because K is superrosy and we have Fact 1.8). Notice also that by Facts 1.8 and 1.9, Conjecture 4 implies Conjecture 2.

Another interesting problem is to classify strongly dependent fields [22, Section 5].

Independently of the questions of A. Hasson and S. Shelah in the NIP context, our approach to attack Conjectures 2 and 3 was to assume that the field in question does not satisfy the conclusion and try to produce a non-trivial definable valuation

(existence of which contradicts rosiness by Fact 1.8). This approach led us to the following conjecture whose assumptions generalize the situations from Conjectures 1, 2, 3 and 4, but whose conclusion is weaker than the conclusions of these conjectures (see Section 4 for explanations). So, one could say that it is a common approximation of these conjectures. Before we formulate our conjecture, let us introduce a certain name for the fields satisfying its assumptions, which was suggested by the referee.

Definition 5 *We say that a field K is radically bounded if for every finite extension L of K and for any natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite and, if $\mathrm{char}(K) = p > 0$ and $f \colon L \to L$ is given by $f(x) = x^p - x$, then the index $[L^+ : f[L]]$ is also finite.*

It is a well-known fact that each perfect, bounded field is radically bounded, whereas the converse is not true as it has been very recently shown in [10]. Fact 1.9 tells us that superrosy fields are radically bounded. In fact, Remark 2.5 and the proof of [24, Theorem 5.6.5] show that they are perfect and bounded, but we will not need it.

Conjecture 6 *Let K be a radically bounded field. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and either algebraically or real closed residue field.*

Our main result is the proof of Conjecture 6 in the positive characteristic case. In fact, we will prove the following theorem.

Theorem 7 *Let K be a radically bounded field. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and, in the case when $\mathrm{char}(K) > 0$, it has algebraically closed residue field.*

By Facts 1.8 and 1.9, one gets the following corollary.

Corollary 8 *Every non-trivial valuation on a superrosy field of positive characteristic has divisible value group and algebraically closed residue field.*

Since infinite NIP fields are closed under Artin-Schreier extensions [13], we also get the following corollary.

Corollary 9 *Suppose K is a field of positive characteristic, satisfying NIP and with the property that for every finite extension L of K and for every natural number $n > 0$ the index $[L^* : (L^*)^n]$ is finite. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and algebraically closed residue field.*

The proof of Theorem 7 relies on [14], where the appropriate results on the existence of non-trivial definable valuations under the presence of certain multiplicative or additive subgroups were established. In contrast, directly from the definition of minimality, we obtain the following variant of Corollary 8 for minimal fields.

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Theorem 10 *Every non-trivial valuation on a minimal field has divisible value group and algebraically closed residue field.*

Recall that the famous Podewski's conjecture predicts that each minimal field is algebraically closed [21]. This conjecture is known to be true in the positive characteristic case [23]; a progress towards a proof in full generality has been made in [17]. As to Theorem 10, Arno Fehm pointed out to me that it can also be deduced from Ralf Shmüker's Ph.D. thesis. We decided to leave our proof, because it is direct and very simple.

Our results tells us that, in various situations, all non-trivial valuations on the field in question have divisible value groups and algebraically closed residue fields. The ultimate goal is to show that the original field (i.e. the residue field with respect to trivial valuation) is algebraically closed [or PAC]. So a questions arises what information about the field in question can be deduced from the information that for all non-trivial valuations the value groups are divisible and the residue fields are algebraically closed. A discussion and some questions about it are included in the last section.

The paper is constructed as follows. First we give prelimnaries concerning rosy theories and valuations, listing all the facts from [14] which are used in the course of the proof of Theorem 7. Section 2 is devoted to the proof of Theorem 7 and some partial results concerning Conjecture 6 in the zero characteristic case. A very short and easy Section 3 is self-contained and yields a proof of Theorem 10. In the last section, we discuss some facts and questions concerning potential applications of our results to prove the original conjectures.

Katharina Dupont is currently working around Conjecture 4 in her Ph.D. project under the supervision of Salma Kuhlmann and in collaboration with Assaf Hasson. She has a different approach than the one presented in this paper (although also based on [14]). A few details on this are mentioned in the last section.

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1 Preliminaries

1.1 Valuations

In this subsection, we list the definitions and facts from [14] which will be useful in this paper. But before that, let us recall the definition of valuation and other basic notions. A good reference for fields with valuations is for example [9].

Definition 1.1 *A valuation on a field K is a surjective map $v \colon K \to \Gamma \cup \{\infty\}$, where $(\Gamma, +)$ is an ordered group and $\Gamma < \infty$, satisfying the following axioms. For all $x, y \in K$:*

1. $v(x) = \infty \Longrightarrow x = 0$,

2. $v(xy) = v(x) + v(y)$,

3. $v(x + y) \geq \min\{v(x), v(y)\}$.

Let v be a valuation on a field K. Define

$$\mathcal{O}_v := \{x \in K : v(x) \geq 0\}.$$

This is a valuation ring, i.e., a subring of K such that for any $x \in K$, either $x \in \mathcal{O}_v$ or $x^{-1} \in \mathcal{O}_v$. We say that the valuation v is trivial if $\Gamma = \{0\}$; equivalently, $\mathcal{O}_v = K$. The group $\mathcal{O}_v^*$ of units of $\mathcal{O}_v$ equals $\{x \in K : v(x) = 0\}$. Finally,

$$\mathcal{M}_v := \{x \in K : v(x) > 0\}$$

is the unique maximal ideal of $\mathcal{O}_v$, and

$$\overline{K}_v := \mathcal{O}_v/\mathcal{M}_v$$

is called the residue field of v.

With a valuation v we associated its valuation ring $\mathcal{O}_v$. Conversely, starting from a valuation ring $\mathcal{O}$, one can define a valuation v so that $\mathcal{O}_v = \mathcal{O}$, and one can do it in such a way that both operations are inverses of each other (after the appropriate identification of value groups). Namely, having a valuation ring $\mathcal{O}$, we define $\Gamma := K^*/\mathcal{O}^*$ with $x\mathcal{O}^* + y\mathcal{O}^* := xy\mathcal{O}^*$, and we order it by $x\mathcal{O}^* \leq y\mathcal{O}^* \iff y/x \in \mathcal{O}$; then we define a valuation v by $v(x) = x\mathcal{O}^* \in \Gamma$.

Definition 1.2 *We say that a valuation $v : K \to \Gamma \cup \{\infty\}$ on a field K (possibly with an additional structure) is definable if the ordered group Γ is interpretable in K and after the interpretation of Γ in K, $\mathrm{graph}(v)$ is definable. By the above discussion, this is equivalent to the fact that $\mathcal{O}_v$ is definable in K.*

If $K \subseteq L$ is a field extension and w is a valuation on L, we say that w extends v if $w{\restriction}_K = v$ (after an isomorphic embedding of the value group of v into the value group of w); equivalently, $\mathcal{O}_w \cap K = \mathcal{O}_v$. In such a situation, we write $(K, \mathcal{O}_v) \subseteq (L, \mathcal{O}_w)$. Let Γ_v be the value group of v and Γ_w the value group of w. Then Γ_v can be treated as a subgroup of Γ_w, and $e(\mathcal{O}_w/\mathcal{O}_v) := [\Gamma_w : \Gamma_v]$ is called the ramification index of the extension $(K, \mathcal{O}_v) \subseteq (L, \mathcal{O}_w)$. Similarly, since $\mathcal{M}_w \cap \mathcal{O}_v = \mathcal{M}_v$, we get that $\overline{K}_v = \mathcal{O}_v/\mathcal{M}_v \hookrightarrow \mathcal{O}_w/\mathcal{M}_w = \overline{L}_w$, and $f(\mathcal{O}_w/\mathcal{O}_v) := [\overline{L}_w : \overline{K}_v]$ is called the residue degree of the extension $(K, \mathcal{O}_v) \subseteq (L, \mathcal{O}_w)$.

Fact 1.3 *Whenever $(K, \mathcal{O}_v) \subseteq (L, \mathcal{O}_w)$ with $n := [L : K]$ being finite, one has the following inequality $e(\mathcal{O}_w : \mathcal{O}_v)f(\mathcal{O}_w : \mathcal{O}_v) \leq n$.*

When v and w are two valuations on the same field K, we say that w is a coarsening of v if $\mathcal{O}_v \subseteq \mathcal{O}_w$. In such a situation, $\mathcal{M}_w \subseteq \mathcal{M}_v$ is a prime ideal of $\mathcal{O}_v$. In this way, one gets a 1-1 correspondence between overrings of $\mathcal{O}_v$ and prime ideals

of $\mathcal{O}_v$. Going further, one gets a 1-1 correspondence between this set of prime ideals and the set of convex subgroups of Γ_v (the value group of v):

$$\Delta \;\mapsto\; p_\Delta := \{x \in K : v(x) > \delta \text{ for all } \delta \in \Delta\}$$
$$p \;\mapsto\; \Delta_p := \{\gamma \in \Gamma : \gamma, -\gamma < v(x) \text{ for all } x \in p\}.$$

For details on this, see [9, Chapter 2.3]. Let us add that two valuations v and w on K are said to be comparable if $\mathcal{O}_v \subseteq \mathcal{O}_w$ or $\mathcal{O}_w \subseteq \mathcal{O}_v$.

From the model-theoretic perspective, an important question is when there exists a non-trivial definable valuation on a given field K. A deep insight into this question is provided in [14]. In particular, with an additive [resp. multiplicative] subgroup T the author associates a certain valuation ring, denoted by $\mathcal{O}_T$, and he gives a complete characterization of when $\mathcal{O}_T$ is first order definable in $(K, +, \cdot, 0, 1, T)$. Here, we recall some definitions and results from [14] which we will use later.

Definition 1.4 *Let v be a valuation on K and T an additive [resp. multiplicative] subgroup.*

1. *v is compatible with T if $\mathcal{M}_v \subseteq T$ [resp. $1 + \mathcal{M}_v \subseteq T$].*

2. *v is weakly compatible with T if $\mathcal{A} \subseteq T$ [resp. $1 + \mathcal{A} \subseteq T$] for some $\mathcal{O}_v$-ideal $\mathcal{A}$ with $\sqrt{A} = \mathcal{M}_v$.*

3. *v is coarsely compatible with T if it is weakly compatible with T and there is no proper coarsening w of v such that $\mathcal{O}_w^* \subseteq T$.*

Fact 1.5 *[14, Lemma 1.2] Let v be a valuation on K. If either T is a multiplicative subgroup such that for some $n \in \omega \setminus \{0\}$, $(K^*)^n \subseteq T$ and we have $(n, \mathrm{char}(\overline{K}_v)) = 1$ or $\mathrm{char}(\overline{K}_v) = 0$, or T is an additive subgroup, $\mathrm{char}(K) = p$ and $\{x^p - x : x \in K\} \subseteq T$, then v is (fully) compatible with T if and only if v is weakly compatible with T.*

Fact 1.6 *[14, Proposition 1.4] For an additive [resp. multiplicative] subgroup T of a field K, any two coarsly compatible valuations are comparable, and there is a unique finest coarsly compatible valuation ring of K which we will denote by $\mathcal{O}_T$. Moreover, $\mathcal{O}_T$ is non-trivial, whenever T is proper (i.e., $T \neq K$ [resp. $T \neq K^*$]) and admits some non-trivial weakly compatible valuation.*

The author concludes that for any additive [resp. multiplicative] subgroup T of a field K exactly one of the following possibilities holds:

- **groups case**: *there is a valuation v on K such that $\mathcal{O}_v^* \subseteq T$.* In this case, $\mathcal{O}_T$ is the only coarsely compatible valuation ring with this property, and all weakly compatible valuations are fully compatible; moreover, all valuations w such that $\mathcal{O}_w^* \subseteq T$ are refinements of $\mathcal{O}_T$.

- **weak case**: *there is a weakly, but not fully compatible valuation on K.*
  In this case, $\mathcal{O}_T$ is the only valuation ring with this property; the weakly compatible valuations are the coarsenings of $\mathcal{O}_T$; there is no valuation v with $\mathcal{O}_v^* \subseteq T$.

- **residue case**: *all weakly compatible valuations are fully compatible, and there is no valuation v with $\mathcal{O}_v^* \subseteq T$.*
  In this case, $\mathcal{O}_T$ is the finest fully compatible valuation ring.

Now, we are going to recall [14, Theorem 2.5], which will be the main tool in this paper. This is a complete characterization of when, for a given additive [resp. multiplicative] subgroup T of a filed K, the ring $\mathcal{O}_T$ is definable in the language $\mathcal{L} := \{+, \cdot, 0, 1, T\}$. In fact, in our applications we will only use the positive part of this characterization (namely the cases when one has definability). Before we formulate the full characterization, we should emphasise that here by definability in $\mathcal{L}$ we do not just mean definability in the structure $(K, +, \cdot, 0, 1, T)$, but the existence of a formula $\varphi(x)$ in $\mathcal{L}$ which defines $\mathcal{O}_{T'}$ in every model $(K', +, \cdot, 0, 1, T') \equiv (K, +, \cdot, 0, 1, T)$.

Fact 1.7 *[14, Theorem 2.5] Let K be a field with an additive [resp. multiplicative] subgroup T. Denote by $\mathcal{M}_T$ the maximal ideal of $\mathcal{O}_T$, and by $\overline{T}$ the subgroup induced by T on the residue field of $\mathcal{O}_T$. Then $\mathcal{O}_T$ is definable in $\mathcal{L} := \{+, \cdot, 0, 1, T\}$ in the following cases:*

	$T \leq (K, +)$	$T \leq K^*$
group case	*iff $\mathcal{O}_T$ is discrete or $(\forall x \in \mathcal{M}_T)(x^{-1}\mathcal{O}_T \not\subseteq T)$*	*always*
weak case	*iff $\mathcal{O}_T$ is discrete*	*iff $\mathcal{O}_T$ is discrete*
residue case	*always*	*iff $\overline{T}$ is no ordering*

1.2 Rosy theories

In this paper, we will only need two properties of rosy groups and fields. Although they are a folklore, for the reader's convenience we recall fundamental definitions concerning rosiness and we give proofs of these two properties.

Fact 1.8 *Let K be a rosy field. Then there is no non-trivial definable valuation on K.*

Fact 1.9 *Let G be a commutative superrosy group. Then for every natural number $n > 0$, if $G[n] := \{g \in G : g^n = e\}$ is finite, the index $[G : G^n]$ is also finite, where G^n denotes the subgroup consiting of n-th powers.*

In particular, if K is a superrosy field, then for every $n > 0$ the index $[K^* : (K^*)^n]$ is finite and similarly, if $char(K) = p \neq 0$, then the image of the function $f \colon K \to K$ defined by $f(x) = x^p - x$ is a subgroup of finite index in K^+. Since any finite extension of an elementary extension of K is also superrosy, this holds for all finite extensions of any elementary extension of K, too. Thus, superrosy fields are radically bounded.

For details on rosy theories the reader is referred to [1, 7, 20], and on rosy groups to [6]. More information about rosy groups and fields can be found in [15, 16].

In this subsection, we work in $\mathfrak{C}^{eq}$ where $\mathfrak{C}$ is a monster model of a theory T in a language $\mathcal{L}$.

A motivation to consider rosy theories is the fact that in a sense it is the largest class of theories which allows the application of techniques from stability theory, especially of basic forking calculus. This class contains stable and more generally simple theories as well as o-minimal theories. The definition of rosiness which justifies what we have just said is the following: T is rosy if there is a ternary relation $\underset{}{\overset{*}{\bigcup}}$ on small subsets of $\mathfrak{C}^{eq}$ satisfying all the basic properties of forking independence in simple theories except for the Independence Theorem. Such a relation will be called an independence relation. There is a concrete, particularly useful independence relation in rosy theories, called þ-independence, which we are going to define now.

A formula $\delta(x, a)$ strongly divides over A if a is not algebraic over A and the set of formulas $\{\delta(x, a')\}_{a' \models \mathrm{tp}(a/A)}$ is k-inconsistent for some $k \in \mathbb{N}$. We say that $\delta(x, a)$ þ-divides over A if we can find some tuple c such that $\delta(x, a)$ strongly divides over Ac. A formula þ-forks over A if it implies a (finite) disjunction of formulas which þ-divide over A.

We say that a type $p(x)$ þ-divides over A if there is a formula implied by $p(x)$ which þ-divides over A; þ-forking is similarly defined. We say that a is þ-independent from b over A, denoted $a \underset{A}{\overset{þ}{\bigcup}} b$, if $\mathrm{tp}(a/Ab)$ does not þ-fork over A.

In rosy theories, þ-independence is the weakest independence relation in the sense that $a \underset{C}{\overset{*}{\bigcup}} b$ implies $a \underset{C}{\overset{þ}{\bigcup}} b$ for any independence relation $\underset{}{\overset{*}{\bigcup}}$.

By a rosy group [or field] we mean a group [or field], possibly with an additional structure, whose theory is rosy.

Rosy theories can be also defined by means of local þ-ranks.

Definition 1.10 *Given a formula $\psi(x)$, a finite set Φ of formulas with object variables x and parameter variables y, a finite set of formulas Θ in variables y, z, and natural number $k > 0$, we define the $þ_{\Phi,\Theta,k}$-rank of ψ inductively as follows:*

1. *$þ_{\Phi,\Theta,k}(\psi) \geq 0$ if ψ is consistent.*

2. *For λ a limit ordinal, $þ_{\Phi,\Theta,k}(\psi) \geq \lambda$ if $þ_{\Phi,\Theta,k}(\psi) \geq \alpha$ for all $\alpha < \lambda$.*

3. *$þ_{\Phi,\Theta,k}(\psi) \geq \alpha + 1$ if there is $\varphi \in \Phi$, some $\theta(y; z) \in \Theta$ and parameter c such that*

   (a) $þ_{\Phi,\Theta,k}(\psi(x) \wedge \varphi(x; a)) \geq \alpha$ for infinitely many $a \models \theta(y; c)$, and

   (b) $\{\varphi(x; a)\}_{a \models \theta(y;c)}$ is k−inconsistent.

Given a (partial) type $\pi(x)$ we define $þ_{\Phi,\Theta,k}(\pi(x))$ to be the minimum of $þ_{\Phi,\Theta,k}(\psi)$ for $\psi \in \pi(x)$.

Recall that a theory is rosy if and only if for each ψ, Φ, Θ, k as above, the local thorn rank $\text{þ}_{\Phi,\Theta,k}(\psi)$ is finite. One could prove Fact 1.8 using this characterization. However, one can give an immediate proof using another characterization of rosiness in terms of the so-called equivalence ranks considered in [7, Section 5].

Definition 1.11 *Let $\pi(x)$ be a partial type, and let Δ be a finite set of formulas in variables x, y, z. Define eq-rk$_\Delta(\pi(x))$ as follows:*

1. *eq-rk$_\Delta(\pi(x)) \geq 0$ if $\pi(x)$ is consistent.*

2. *For λ a limit ordinal, eq-rk$_\Delta(\pi(x)) \geq \lambda$ if eq-rk$_\Delta(\pi(x)) \geq \alpha$ for all $\alpha < \lambda$.*

3. *eq-rk$_\Delta(\pi(x)) \geq \alpha + 1$ if there is some equivalence relation $E(x, y)$ defined by $\delta(x, y, c)$ with $\delta(x, y, z) \in \Delta$ and $c \in \mathfrak{C}^{eq}$, and there are representatives b_i, $i < \omega$, of different equivalence classes, such that eq-rk$_\Delta(\pi(x) \wedge E(x, b_i)) \geq \alpha$.*

From [7, Section 5], we know that T is rosy if and only if for every Δ and $\pi(x)$ as above, eq-rk$_\Delta(\pi(x))$ is finite.

Proof of Fact 1.8. Suppose for a contradiction that there is a non-trivial definable valuation v on the rosy field K. We can assume K is a monster model. For x, y from the sort of K and z from the sort of Γ consider the formula

$$\delta(x, y, z) = (v(x - y) \geq z).$$

Then, for $\gamma \in \Gamma$ the formula $\delta(x, y, \gamma)$ defines an equivalence relation on K which we denote by $E_\gamma(x, y)$. Put $\Delta = \{\delta(x, y, z)\}$.

It is enough to show that eq-rk$_\Delta(\delta(x, b, \gamma)) \geq n$ for all $n \in \omega$, $\gamma \in \Gamma$ and $b \in K$, because then eq-rk$_\Delta(\delta(x, b, \gamma))$ are infinite, so K is not rosy.

We argue by induction on n. The case $n = 0$ is trivial. Suppose eq-rk$_\Delta(\delta(x, b, \gamma)) \geq n$ for all $n \in \omega$, $\gamma \in \Gamma$ and $b \in K$. Consider any $\gamma \in \Gamma$ and $b \in K$. By saturation, there is $\gamma' \in \Gamma$ such that $\gamma' > \gamma$ and $E_{\gamma'}(x, y)$ refines $[b]_{E_\gamma}$ into infinitely many classes, say with representatives b_i, $i \in \omega$. Then, taking $c := \gamma'$ in the definition of eq-rk$_\Delta$, we get eq-rk$_\Delta(\delta(x, b, \gamma)) \geq n + 1$. ∎

Using $\underset{\smile}{\downarrow}^\text{þ}$, we define U$^\text{þ}$-rank in the same way as U-rank is defined in stable theories by means of $\underset{\smile}{\downarrow}$, namely U$^\text{þ}$ is a unique function from the collection of all complete types to the ordinals together with ∞ with the property that for any ordinal α, U$^\text{þ}(p) \geq \alpha + 1$ if and only if there is some tuple a and some type $q \in S(Aa)$ such that $q \supset p$, U$^\text{þ}(q) \geq \alpha$ and q þ-forks over A. U$^\text{þ}$-rank in rosy theories has most of the nice properties that U-rank has in stable theories, e.g. it satisfies Lascar Inequalities:

$$\text{U}^\text{þ}(a/b, A) + \text{U}^\text{þ}(b/A) \leq \text{U}^\text{þ}(a, b/A) \leq \text{U}^\text{þ}(a/b, A) \oplus \text{U}^\text{þ}(b/A).$$

Assume T is rosy. If D is an A-definable set, then U$^\text{þ}(D) := \sup\{\text{U}^\text{þ}(d/A) : d \in D\}$. Of course, if this supremum is finite, then it is just the maximum. It turns out that if

D is a definable group, then the supremum is also attained [6, Remark 1.20]. For D a definable set, $\mathrm{U}^\flat(D) = 0$ if and only if D is finite. There are also Lascar inequalities for groups: For definable groups $H \leq G$ we have

$$\mathrm{U}^\flat(H) + \mathrm{U}^\flat(G/H) \leq \mathrm{U}^\flat(G) \leq \mathrm{U}^\flat(H) \oplus \mathrm{U}^\flat(G/H).$$

We say that T is superrosy if $\mathrm{U}^\flat(p) < \infty$ for every type p; a group [or field] is superrosy if its theory is superrosy.

Proof of Fact 1.9. We only show the first part of the fact. Let $H = G^n$. We claim that $\mathrm{U}^\flat(G) = \mathrm{U}^\flat(H)$. To see this, take $g \in G$ with $\mathrm{U}^\flat(g/\emptyset) = \mathrm{U}^\flat(G)$. Of course, $g^n \in \mathrm{acl}(g)$, so $\mathrm{U}^\flat(g^n/g) = 0$. Since $G[n]$ is finite, we get $g \in \mathrm{acl}(g^n)$, and so $\mathrm{U}^\flat(g/g^n) = 0$. Thus, by Lascar Inequalities, we get $\mathrm{U}^\flat(g^n) = \mathrm{U}^\flat(g)$, so $\mathrm{U}^\flat(H) = \mathrm{U}^\flat(G)$. Using Lascar Inequalities for groups, we immediately conclude that $\mathrm{U}^\flat(G/H) = 0$, so G/H is finite. ∎

Definition 1.12 *We say that a theory T has the NIP if there is no formula $\varphi(x,y)$ and sequence $\langle a_i \rangle_{i<\omega}$ such that for every $w \subseteq \omega$ there is b_w such that $\models \varphi(a_i, b_w)$ iff $i \in w$.*

The main result of [13] says that an NIP field K has no Artin-Schreier extensions, i.e., if $\mathrm{char}(K) = p > 0$, then the function $x \mapsto x^p - x$ from K to K is surjective. The proof in general uses some algebraic geometry, but assuming that the image of the function $x \mapsto x^p - x$ is of finite index, this is an immediate consequence of the existence of $(K^+)^{00}$ (i.e., of the smallest type-definable subgroup of K^+ of bounded index (wlog we assume here that K is a monster model)). Indeed, one easily checks that $(K^+)^{00}$ is a non-trivial ideal of K, so $(K^+)^{00} = K$. Since the image of $x \mapsto x^p - x$ is a definable subgroup of K^+ of finite index, we get that it contains $(K^+)^{00} = K$, so it is equal to K.

2 Radically bounded fields

This section is devoted to the proof of Theorem 7 and some observations concerning Conjecture 6 in the zero characteristic case.

We start from some preparatory observations.

Remark 2.1 *If $K \subseteq L$ is an algebraic field extension and v is a non-trivial valuation on L, then $v{\restriction}_K$ is also non-trivial.*

Proof. Suppose for a contradiction that $v{\restriction}_K$ is trivial. Consider any $a \in L$ with $v(a) > 0$. Let $P(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0 \in K[x]$ be the minimal monic polynomial of a over K. Then $v(a^n) > v(a_{n-1}a^{n-1}) > \ldots > v(a_0) = 0$, so $\infty = v(0) = v(P(a)) = v(a_0) = 0$, a contradiction. ∎

Remark 2.2 *Let L be a finite extension of a field K. Suppose that any non-trivial valuation on L has the property that its residue field is algebraically closed. Then the residue field of any non-trivial valuation on K is either algebraically or real closed.*

Proof. Consider any non-trivial valuation v on K. By Chevalley's Extension Theorem, v has an extension to a (non-trivial) valuation w on L. By assumption, $\overline{L}_w$ is algebraically closed. Since, by Fact 1.3, $\overline{L}_w$ is a finite extension of $\overline{K}_v$, the conclusion follows. ∎

Lemma 2.3 *Let v be a valuation on a field K. Let F be a finite extension of the residue field $\overline{K}_v$. Then there is a finite extension L of K and an extension w of v to L such that the field F is isomorphic over $\overline{K}_v$ to the residue field $\overline{L}_w$.*

Proof. It is enough to prove it for 1-generated extensions. So, let $F = \overline{K}_v(\overline{\alpha})$ for some $\overline{\alpha} \in F$. Choose $P(x) \in \mathcal{O}_v[x]$ a monic polynomial such that $\overline{P}(x)$ is the minimal monic polynomial of $\overline{\alpha}$ over $\overline{K}_v$, where $\overline{P}(x) \in \overline{K}_v[x]$ denotes the polynomial obtained from $P(x)$ by reducing the coefficients modulo $\mathcal{M}_v$. Consider a root β of $P(x)$, put $L := K(\beta)$ and take any extension of v to a valuation w on L. Since the coefficients of $P(x)$ are in $\mathcal{O}_v$ with the leading coefficient equal to 1, one easily gets that $w(\beta) \geq 0$, so $\beta \in \mathcal{O}_w$.

Now, for $\overline{\beta} := \beta + \mathcal{M}_w \in \overline{L}_w$, we have that $\overline{P}(\overline{\beta}) = 0$. Since $\overline{P}$ is irreducible, we get
$$[\overline{L}_w : \overline{K}_v] \geq [\overline{K}_v(\overline{\beta}) : \overline{K}_v] = \deg \overline{P} = \deg P \geq [L : K],$$
so we have everywhere equalities by Fact 1.3. Therefore, $\overline{L}_w = \overline{K}_v(\overline{\beta})$ which is isomorphic to F over $\overline{K}_v$. ∎

The next lemma recalls a standard method of showing that a given field is algebraically closed.

Lemma 2.4 *Let K be a field. If for every finite extension L of K and for every prime number $n > 0$, $(L^*)^n = L^*$ and, in the case when $\mathrm{char}(K) = p > 0$, the function $f : L \to L$ given by $f(x) = x^p - x$ is onto, then K is algebraically closed.*

Proof. By assumption, K is perfect. If it is not algebraically closed, then it has a proper Galois extension F of minimal degree $k > 1$. There is an intermediate field L with $\mathrm{Gal}(F/L) \cong \mathbb{Z}_q$ for some prime number q.

If $q = p = \mathrm{char}(K)$, then Galois theory tells us that F is the splitting field over L of a polynomial of the form $x^p - x - a$ for some $a \in L$, but, by assumption, this polynomial has at least one zero in L, so all its zeros are in L, a contradiction.

Assume $q \neq \mathrm{char}(K)$. Since $q \leq k$ and for a primitive q-th root of unity ζ_q the field $K(\zeta_q)$ is a Galois extension of K of degree less than q, by the choice of k, we conclude that $\zeta_q \in K$. So $\zeta_q \in L$. By Galois theory, we conclude that F is a splitting field over L of a polynomial of the form $x^q - a$ for some $a \in L$. By assumption, this polynomial has a zero in L, so all its zeros are in L, a contradiction. ∎

Remark 2.5 *Let K be an infinite field of characteristic $p > 0$. Then either $K^p = K$, or the index $[K^* : (K^*)^p]$ of multiplicative groups is infinite and the index $[K^+ : (K^+)^p]$ of additive groups is also infinite. In particular, if the index $[K^* : (K^*)^p]$ is finite, then K is perfect.*

Proof. Suppose $K^p \neq K$, and let $b \in K$ be such that b and 1 are linearly independent over K^p. Then $1 + k^p b$ for $k \in K$ are in different cosets modulo $(K^*)^p$. So the index $[K^* : (K^*)^p]$ is infinite. Similarly, the elements $k^p b$ for $k \in K$ are in different additive cosets modulo $(K^+)^p$, so the index $[K^+ : (K^+)^p]$ is also infinite ∎

Proof of Theorem 7. Suppose there is no non-trivial definable valuation on K.

Part 1. *The value group of any non-trivial valuation v on K is divisible.*

Proof. Suppose for a contradiction that some $\gamma \in \Gamma_v$ is not divisible by n. Put

$$T = \mathcal{O}_v^* \cdot (K^*)^n.$$

Claim 1 *T is a proper, definable subgroup of K^* such that $\mathcal{O}_v^* \leq T$ and $(K^*)^n \leq T$.*

Proof of Claim 1. Since $(K^*)^n$ is a finite index subgroup of K^*, we get that T is a union of finitely many cosets of $(K^*)^n$, so T is definable and contains $(K^*)^n$. The fact that $\mathcal{O}_v^* \leq T$ follows directly from the definition of T. To see that T is proper, suppose for a contradiction that $T = K^*$. Then there is $a \in T$ such that $v(a) = \gamma$. By the definition of T, we can write $a = r \cdot k^n$ for some $r \in \mathcal{O}_v^*$ and $k \in K^*$. Then $\gamma = v(a) = nv(k) \in n\Gamma_v$, a contradiction. ∎

By Claim 1, we know that T is proper and $1 + \mathcal{M}_v \subseteq \mathcal{O}_v^* \leq T$, so v is fully compatible with T. By Fact 1.6, we conclude that $\mathcal{O}_T$ is non-trivial. We also get that we are in the group case, so Fact 1.7 tells us that $\mathcal{O}_T$ is definable in $(K, +, \cdot, 0, 1, T)$, but T is definable in K, so $\mathcal{O}_T$ is definable in K. This is a contradiction, which completes the proof of Part 1. ∎

Part 2. *Assuming that $\mathrm{char}(K) = p > 0$, the residue field of any non-trivial valuation v on K is algebraically closed.*

Proof. By Lemmas 2.3 and 2.4, we will be done if we prove the following two claims.

Claim 2 *For every finite extension L of K, for every extension w of v to a valuation on L, and for every prime number n, one has $(\overline{L}_w^*)^n = \overline{L}_w^*$.*

Claim 3 *For every finite extension L of K and for every extension w of v to a valuation on L one has that the function $f \colon \overline{L}_w \to \overline{L}_w$ given by $f(x) = x^p - x$ is onto.*

Proof of Claim 2. First, notice that by Remark 2.1, we can assume that $L = K$. Indeed, since L is definable in K (living in some Cartesian power of K), by Remark

2.1, we get that there is no non-trivial definable valuation on L; it is also clear that L is radically bounded.

Consider the case $\boldsymbol{n \neq p}$.

Subclaim $K^n(1 + \mathcal{M}_v) = K$

Proof of the subclaim. Suppose it is not true. Then $T := (K^*)^n(1 + \mathcal{M}_v)$ is a proper subgroup of K^*, and v is fully compatible with T. By Fact 1.6, $\mathcal{O}_T$ is non-trivial.

Since $[K^* : (K^*)^n]$ is finite, we see that T is definable, so there is no non-trivial valuation on K definable in $(K, +, \cdot, 0, 1, T)$.

Let $\overline{K}_T$ be the residue field corresponding to the valuation ring $\mathcal{O}_T$. Since $(K^*)^n \leq T$ and $(n, \mathrm{char}(\overline{K}_T)) = 1$, Fact 1.5 gives us that we are either in the group case or in the residue case (if we were in the weak case, then $\mathcal{O}_T$ would be weakly but not fully compatible with T which would contradict Fact 1.5). By Fact 1.7, in the group case, $\mathcal{O}_T$ is definable in $(K, +, \cdot, 0, 1, T)$ which is impossible, and, in the residue case, either $\mathcal{O}_T$ is definable in $(K, +, \cdot, 0, 1, T)$ which is impossible or $\overline{T}$ is the positive cone of an ordering on $\overline{K}_T$. However, the last thing is also impossible as $\mathrm{char}(\overline{K}_T) = p > 0$. ∎

By the subclaim, $\overline{K}_v^n = \overline{K}_v$. Indeed, for any $a \in \mathcal{O}_v$, $a = k^n(1 + m)$ for some $k \in K$ and $m \in \mathcal{M}_v$. Since $1 + m \in \mathcal{O}_v^*$, we get that $k \in \mathcal{O}_v$, so $\overline{a} = \overline{k}^n \in \overline{K}_v^n$.

Now, consider the case $\boldsymbol{n = p}$. Suppose for a contradiction that $\overline{K}_v^p$ is a proper subfield of $\overline{K}_v$. As we have already proved that $\overline{K}_v^m = \overline{K}_v$ whenever $(m, \mathrm{char}(\overline{K}_v)) = 1$, we see that $\overline{K}_v$ is infinite. Thus, Remark 2.5 implies that $[\overline{K}_v^* : (\overline{K}_v^*)^p]$ is infinite. Moreover, if $\overline{a}\overline{K}_v^p \neq \overline{b}\overline{K}_v^p$ for some $a, b \in \mathcal{O}_v$, then $aK^p \neq bK^p$, because otherwise either $ak^p = b$ or $a = bk^p$ for some $k \in \mathcal{O}_v$, and so either $\overline{a}\overline{k}^p = \overline{b}$ or $\overline{a} = \overline{b}\overline{k}^p$, a contradiction. This implies that $[K^* : (K^*)^p]$ is infinite, which contradicts the assumption that K is radically bounded. ∎

Proof of Claim 3. As in the proof of Claim 2, by Remark 2.1, we can assume that $K = L$. Let $F \colon K \to K$ be given by $F(x) = x^p - x$.

Subclaim $F[K] + \mathcal{M}_v = K$.

Proof of the subclaim. Suppose it is not true. Then $T := F[K] + \mathcal{M}_v$ is a proper subgroup of K^+, and v is fully compatible with T. By Fact 1.6, $\mathcal{O}_T$ is non-trivial. As $[K^+ : F[K]]$ is finite, T is definable in K, so there is no non-trivial valuation on K definable in $(K, +, \cdot, 0, 1, T)$.

Since $F[K] \leq T$, Fact 1.5 ensures that we are either in the group case or in the residue case. In the residue case, $\mathcal{O}_T$ is definable by Fact 1.7, a contradiction. In the group case, since we know that $\mathcal{O}_T$ is not definable, Fact 1.7 yields some $x \in \mathcal{M}_T$ with $x^{-1}\mathcal{O}_T \subseteq T$.

Now, we will adopt the argument from the proof of [14, Theorem 3.1]. For the reader's convenience, we give all the details.

Let $\mathcal{A}_T$ be the largest fractional $\mathcal{O}_T$-ideal contained in T. We aim at defining

a certain fractional $\mathcal{O}_T$-ideal $\mathcal{A}_T^\alpha$ which will properly contain $\mathcal{A}_T$ and which will be contained in T. This will contradict the choice of $\mathcal{A}_T$.

Denote by v_T and by Γ_T the valuation and the value group corresponding to $\mathcal{O}_T$. For a real number α and $\gamma_1, \gamma_2 \in \Gamma_T$, by $\gamma_1 \geq \alpha\gamma_2$ we mean that $\gamma_1 \geq r\gamma_2$ for all rationals $r \leq \alpha$ if $\gamma_2 \geq 0$, or for all rationals $r \geq \alpha$ if $\gamma_2 < 0$. This definition makes sense since Γ_T is divisible by Part 1 of the proof. By $\gamma_1 < \alpha\gamma_2$ we mean the negation of $\gamma_1 \geq \alpha\gamma_2$. For $\alpha > 1$ define

$$\mathcal{A}_T^\alpha := \{x \in K : v_T(x) \geq \alpha v_T(y) \text{ for some } y \in \mathcal{A}_T\}.$$

This is, of course, a fractional $\mathcal{O}_T$-ideal. Since $x^{-1}\mathcal{O}_T \subseteq T$ and $x \in \mathcal{M}_T$, we have that $\mathcal{A}_T$ properly contains $\mathcal{O}_T$. This easily implies that $\mathcal{A}_T \subseteq \mathcal{A}_T^\alpha$. Now, we check that this inclusion is proper.

Take a natural number n such that $\alpha > 1 + \frac{1}{n}$. There is $y \in \mathcal{A}_T \setminus \mathcal{O}_T$ such that $(1 + \frac{1}{n})v_T(y) \notin v_T(\mathcal{A}_T)$, as otherwise one easily gets that for all $y \in \mathcal{A}_T \setminus \mathcal{O}_T$, $y\mathcal{O}_T$ generates a valuation ring properly containing $\mathcal{O}_T$ and contained in T, which contradicts the fact that v_T is coarsely compatible with T. Choose $z \in K$ with $v_T(z) = \frac{1}{n}v_T(y)$. Then $v_T(zy) = (1 + \frac{1}{n})v_T(y) \geq \alpha v_T(y)$, so $zy \in \mathcal{A}_T^\alpha \setminus \mathcal{A}_T$.

The proof of the subclaim will be completed if we show that $\mathcal{A}_T^\alpha \subseteq T$ for $\alpha \in (1, 2 - \frac{1}{p})$. It is enough to prove that every element $t \in \mathcal{A}_T^\alpha \setminus \mathcal{A}_T$ belongs to T. We have $v_T(t) \geq \alpha v_T(y)$ for some $y \in \mathcal{A}_T \setminus \mathcal{O}_T$. Since $t \notin \mathcal{A}_T$, we immediately get $v_T(ty^{-1}) = v_T(t) - v_T(y) < 0$. Therefore, $ty^{-1} \in K \setminus \mathcal{O}_T$. Thus, by Remark 2.5, $ty^{-1} = a^p$ for some $a \in K \setminus \mathcal{O}_T$, and so $ta^{-p} = y \in \mathcal{A}_T \setminus \mathcal{O}_T \subseteq T \setminus \mathcal{O}_T$, hence $ta^{-p} = b^p - b + m$ for some $b \in K \setminus \mathcal{O}_T$ and $m \in \mathcal{M}_v \subseteq \mathcal{O}_T$. (To see the last inclusion, one should consider two cases. The first one is when $\mathcal{O}_v \nsubseteq T$. Since T is an additive subgroup, this implies that v is coarsely compatible with T, so $\mathcal{O}_T \subseteq \mathcal{O}_v$, hence $\mathcal{M}_v \subseteq \mathcal{O}_T$. The second case is when $\mathcal{O}_v \subseteq T$. Then we are in the group case, so $\mathcal{O}_v \subseteq \mathcal{O}_T$.) As $\alpha < 2 - \frac{1}{p}$, we obtain

$$
\begin{aligned}
v_T(a^p b) &= v_T(a^p) + v_T(b) = v_T(ty^{-1}) + v_T(b) = v_T(t) - v_T(y) + \tfrac{1}{p}v_T(y) \\
&\geq \left(\alpha - 1 + \tfrac{1}{p}\right)v_T(y) > v_T(y).
\end{aligned}
$$

Therefore, $v_T(ab) > v_T(a^p b) > v_T(y)$ and $v_T(a^p m) > v_T(a^p b) > v_T(y)$. Hence, $ab, a^p b, a^p m \in \mathcal{A}_T \subseteq T$, and so $t = ((ab)^p - ab) + ab - a^p b + a^p m \in T$. ■

By the subclaim, $f[\overline{L}_v] = \overline{L}_v$. Indeed, for any $a \in \mathcal{O}_v$, $a = k^p - k + m$ for some $k \in K$ and $m \in \mathcal{M}_v$. Then $k \in \mathcal{O}_v$, so $\overline{a} = \overline{k}^p - \overline{k}$. ■

So, the proof of Part 2 and of the whole theorem has been completed. ■

As was pointed out in the introduction, Corollary 8 follows from Theorem 7 and Facts 1.8 and 1.9, and Corollary 9 follows from Theorem 7 and the fact that NIP fields are closed under Artin-Schreier extensions.

Proposition 2.6 *Let K be a radically bounded field of characteristic zero. Then either there is a non-trivial definable valuation on K, or for every non-trivial valuation v on K:*

1. *if $\operatorname{char}(\overline{K}_v) > 0$, then $\overline{K}_v^n = \overline{K}_v$ for every $n > 0$,*

2. *if $\operatorname{char}(\overline{K}_v) = 0$ and $\sqrt{-1} \in K$, then there is a prime number p such that for all primes n different from p, $\overline{K}_v^n = \overline{K}_v$.*

Proof. Suppose there is no non-trivial definable valuation on K.

(1) Almost the same argument (based on Facts 1.5, 1.6 and 1.7) as in the proof of Claim 2 in the proof of Theorem 7 works. The only difference is that we show the weaker version of the subclaim which says that $T := (K^*)^n(1 + \mathcal{M}_v) \supseteq \mathcal{O}_v^*$ and which is of course strong enough to finish the proof. The distinction between cases $n \neq p$ and $n = p$ is considered for $p := \operatorname{char}(\overline{K}_v)$. To prove this weaker version of the subclaim, one can apply the same argument as before, noting that since we assume that $\mathcal{O}_v^* \nsubseteq T$, we get $\mathcal{O}_T \subseteq \mathcal{O}_v$, so $\operatorname{char}(\overline{K}_T) = \operatorname{char}(\overline{K}_v) = p$ is finite.

(2) Suppose that for some prime p, $\overline{K}_v^p \neq \overline{K}_v$. Then $T := (K^*)^p(1 + \mathcal{M}_v)$ is a proper, definable subgroup of K^*. If $p \neq \operatorname{char}(\overline{K}_T)$, then we get a contradiction as in the proof of the subclaim in the proof of Claim 2 in Theorem 7, noting that the assumption $\sqrt{-1} \in K$ eliminates the possibility that $\overline{T}$ is the positive cone of an ordering on $\overline{K}_T$. So $p = \operatorname{char}(\overline{K}_T)$.

Consider any prime $n \neq p$. Let $T' := (K^*)^{np}(1 + \mathcal{M}_v)$. Then $T' \leq T$. On the other hand, since T is definable and $\mathcal{O}_T$ is not definable, Fact 1.7 implies that T does not belong to the group case. Thus, we conclude that $\mathcal{O}_T \subseteq \mathcal{O}_{T'}$, and hence $\operatorname{char}(\overline{K}_{T'}) \in \{0, p\}$. However, it is impossible to have $\operatorname{char}(\overline{K}_{T'}) = 0$, as in this case once again one gets a contradiction as in the proof of the subclaim in the proof of Claim 2 in Theorem 7. So $\operatorname{char}(\overline{K}_{T'}) = p$. Our goal is to show that $K^n(1 + \mathcal{M}_v) = K$. If this is not the case, then $T'' := (K^*)^n(1 + \mathcal{M}_v)$ is a proper, definable subgroup of K^*, and once again Facts 1.5, 1.6 and 1.7 yield that $(n, \operatorname{char}(K_{T''})) \neq 1$ and T'' does not belong to the group case. Since $T' \leq T''$, we conclude that $\mathcal{O}_{T''} \subseteq \mathcal{O}_{T'}$, but $\operatorname{char}(\overline{K}_{T'}) = p$, and so we get that $\operatorname{char}(\overline{K}_{T''}) = p$. This contradicts the assumption that n is different from and so relatively prime to p. ∎

If one was able to strengthen the conclusion of Proposition 2.6(2) by showing that $\overline{K}_v^n = \overline{K}_v$ for all primes n, then using Remark 2.1 and Lemmas 2.3 and 2.4, one would get that, under the assumption of Proposition 2.6 (together with $\sqrt{-1} \in K$), either there is a non-trivial definable valuation on K, or for any non-trivial valuation v with $\operatorname{char}(\overline{K}_v) = 0$ the residue field $\overline{K}_v$ is algebraically closed. So, using Remark 2.1 and arguing as in the proof of Remark 2.2, one would also get that the assumption that the field is radically bounded implies that either there is a non-trivial definable valuation on K, or for any non-trivial valuation v with $\operatorname{char}(\overline{K}_v) = 0$ the residue field $\overline{K}_v$ is either algebraically or real closed. In fact, strengthening slightly the definition of radical boundedness, this would imply the full conclusion of Conjecture 6.

Proposition 2.7 *Suppose that the conclusion of Proposition 2.6(2) can be strengthened to '$\overline{K}_v^n = \overline{K}_v$ for all primes n'. Let K be a field whose all elementary extensions are radically bounded. Then either there is a non-trivial definable valuation on K, or every non-trivial valuation on K has divisible value group and either algebraically or real closed residue field.*

Proof. Suppose there is no non-trivial definable valuation on K. Using Remarks 2.1 and 2.2, we can assume that $\sqrt{-1} \in K$. Let v be any non-trivial valuation on K. Divisibility of Γ_v was proved in Theorem 7. By this theorem, it remains to consider the case $\operatorname{char}(K) = 0$. From the discussion above Proposition 2.7, we are done in the case $\operatorname{char}(\overline{K}_v) = 0$. So it remains to consider the case $\operatorname{char}(\overline{K}_v) = p > 0$.

Take a monster model $(\widetilde{K}, \widetilde{\Gamma}, \widetilde{v}) \succ (K, \Gamma_v, v)$. Let $\mathcal{O}$ be a maximal valuation ring in $\widetilde{K}$ containing $\mathcal{O}_{\widetilde{v}}$ and such that $\frac{1}{p} \notin \mathcal{O}$. Let $\mathcal{M}$ be the maximal ideal of $\mathcal{O}$. Then $p \in \mathcal{M}$, so $k := \mathcal{O}/\mathcal{M}$ is of characteristic p.

We claim that there is a non-trivial valuation ring $\mathcal{O}_1$ in $\widetilde{K}$ such that $\mathcal{O} \subsetneq \mathcal{O}_1$. To see this, recall that $\mathcal{O}$ corresponds to the proper convex subgroup $\widetilde{\Gamma}_{\mathcal{O}}$ of $\widetilde{\Gamma}$ defined by

$$\widetilde{\Gamma}_{\mathcal{O}} := \{\gamma \in \widetilde{\Gamma} : (\forall x \in \mathcal{M})(\widetilde{v}(x) > \gamma, -\gamma)\}.$$

So, there is $\gamma > \widetilde{\Gamma}_{\mathcal{O}}$. Put $\widetilde{\Gamma}_1 := \bigcup_n (-n\gamma, n\gamma)$ a convex subgroup of $\widetilde{\Gamma}$ properly containing $\widetilde{\Gamma}_{\mathcal{O}}$. By saturation, $\widetilde{\Gamma}_1 \neq \widetilde{\Gamma}$. So, $\widetilde{\Gamma}_1 = \widetilde{\Gamma}_{\mathcal{O}_1}$ for some non-trivial valuation ring $\mathcal{O}_1 \supsetneq \mathcal{O}$.

Since $\mathcal{O} \subsetneq \mathcal{O}_1$, we have that $\frac{1}{p} \in \mathcal{O}_1$, hence the residue field $k_1 := \mathcal{O}_1/\mathcal{M}_1$ (where $\mathcal{M}_1$ is the maximal ideal of $\mathcal{O}_1$) is of characteristic 0. So k_1 is algebraically closed (as the conclusion of Proposition 2.7 holds in the residue zero characteristic case, also for $\widetilde{K}$ in place of K). As $\mathcal{O}_{\widetilde{v}} \subseteq \mathcal{O}_1$, this implies that the residue field $\overline{\widetilde{K}}_{\widetilde{v}}$ is algebraically closed, and so $\overline{K}_v$ is algebraically closed, too. ∎

By [10], we know that the property 'all elementary extensions of K are radically bounded' is strictly stronger than radical boundedness and strictly weaker that boundedness. By Fact 1.9, superrosy fields satisfy this strengthening of radical boundedness (as was mentioned in the introduction, they are even bounded).

3 Minimal fields

We will prove here Theorem 10. In contrast to Theorem 7, where the proof relies on non-trivial results from [14], the proof of Theorem 10 is a trivial consequence of the definition of minimality.

Recall that a minimal field is an infinite field whose every definable (in one variable) subset is finite or co-finite.

Proof of Theorem 10. Consider any non-trivial valuation v on K.

By minimality, for every natural number $n > 0$, $(K^*)^n = K^*$. (To see this, notice that $(K^*)^n$ is an infinite multiplicative subgroup of K, so it is a co-finite subgroup, so it is everything.) From this, it is clear that Γ_v is divisible.

Consider any monic polynomial $P(x) = x^n + \overline{a}_{n-1} x^{n-1} + \ldots + \overline{a_0} \in \overline{K}_v[x]$ of positive degree n; here $a_0, \ldots, a_{n-1} \in \mathcal{O}_v$. Let $P(x) = x^n + a_{n-1} x^{n-1} + \ldots + a_0$. Since $P(x)$ takes co-finitely many values in K and $\mathcal{M}_v$ is infinite, there exists $a \in K$ such that $P(a) \in \mathcal{M}_v$. Then $a \in \mathcal{O}_v$, as otherwise $v(a) < 0$, and so $v(a_i a^i) = v(a_i) + iv(a) > nv(a) = v(a^n)$ for $i = 0, \ldots, n-1$, which implies that $v(P(a)) = v(a^n) = nv(a) < 0$, a contradiction with the fact that $P(a) \in \mathcal{M}_v$. Therefore, $\overline{P}(\overline{a}) = 0$, i.e., $\overline{P}(x)$ has a root in $\overline{K}_v$. ∎

4 Final comments

We will say that a given field K has Property $(*)$ if every non-trivial valuation on K has divisible value group and algebraically closed residue field. We will say that it has Property $(*-)$ if every non-trivial valuation on K has divisible value group and either algebraically or real closed residue field. By Remark 2.2, a field K has Property $(*-)$ if and only if $K(\sqrt{-1})$ has Property $(*)$.

Having the results and conjectures discussed in this paper, a natural questions arises what can be said about the structure of fields with Property $(*)$ or $(*-)$. In particular, can one deduce from our results Conjectures 1, 2, 3, 4 (at least in positive characteristic) or Podewski's conjecture?

Recall that a field is orderable if it can be equipped with some order making it an ordered field; equivalently, it is formally real (i.e., -1 is not a sum of squares). A field K is PAC (pseudo algebraically closed) if every absolutely irreducible variety over K has a K-rational point. A field K is PRC (pseudo real closed) if every absolutely irreducible variety over K which has an F-rational point in every real closed field F containing K has a K-rational point. With such definitions each PAC field is PRC. We know that if K is an orderable PRC field, then $K(\sqrt{-1})$ is (perfect) PAC [20, Lemma A.1.1.3].

Fact 4.1 *A PRC field is superrosy if and only if it is perfect and bounded.*

Proof. ($\leftarrow$) was proved in [20, Appendix A] for orderable PRC fields and in [12] for PAC fields. To see the converse, note that the same argument as in [24, Theorem 5.6.5] yields boundedness. Finally, suppose for a contradiction that K is not perfect. Then, by Remark 2.5, $[K^* : (K^*)^p]$ is infinite, which contradicts Fact 1.9. ∎

Remark 4.2 *An orderable PRC field has the strict order property, so it is not simple. In particular, Conjecture 3 implies Conjecture 1.*

Proof. Let K be a formally real PRC field. We claim that any element which is a sum of finitely many squares is a sum of two squares. For this, consider any non-zero element $a \in K$ which is a sum of finitely many squares. Then the polynomial

x^2+y^2-a has a zero in any real closed field containing K. Since x^2+y^2-a is absolutely irreducible (e.g. by the Eisenstein criterion) and K is PRC, this polynomial has a zero in K, and so a is a sum of two squares in K.

Now, define the relation $\leq$ on K by

$$x \leq y \iff y - x \text{ is a sum of squares of finitely many elements of } K.$$

We claim that $\leq$ is a partial order with an infinite chain. The fact that $\leq$ is antisymmetric follows from the fact that K is formally real. Transitivity is clear. The existence of an infinite chain can be seen as follows. Take any $a \neq 0$. Then $a^2 < 2a^2 < 3a^2 < \dots$.

Since, by the first paragraph of the proof, $x \leq y$ if and only if $y - x$ is a sum of two squares, we see that $\leq$ is definable. ∎

Fact 4.3 *A non separably and non real closed PRC field does not have NIP. In particular, Conjecture 3 implies Conjecture 2.*

Proof. The main result of [5] says that a non separably closed PAC field does not have NIP. Now, consider a non separably and non real closed PRC field. Suppose for a contradiction that it has NIP. Then $K(\sqrt{-1})$ still has NIP (as it is interpretable in K) and it is PAC but not separably closed, a contradiction. ∎

By [8], perfect PAC fields have Property $(*)$, so, by [20, Lemma A.1.1.3], perfect PRC fields have Property $(*-)$. One could ask whether for infinite fields Property $(*-)$ [or $(*)$] implies that the field in question is PRC (in the non formally real case, PRC can be replaced by PAC). By virtue of our results and Fact 4.3, the positive answer would imply Conjectures 1, 2, 3 and 4 in positive characteristic. Unfortunately the answer is negative, which we briefly explain now. In [8] and [11], the Hasse principle for Brauer groups is considered. It is shown in [8, Theorem 3.4] that if K is a perfect PAC field, then any extension F of K of relative transcendence degree 1 satisfies the Hasse principle for the Brauer groups. On the other hand, it is shown in [8, Theorem 4.1] that whenever K is a perfect field such that the Hasse principle for the Brauer groups holds for all extensions F of K of relative transcendence degree 1, then the field K has Property $(*)$. It was also asked in [8, Question 4.2] whether a non formally real infinite perfect field K such that the Hasse principle for the Brauer groups holds for all extensions F of K of relative transcendence degree 1 is necessarily PAC? In [11], a counter-example was constructed. The authors introduced the class of the so-called weakly PAC fields, they proved that weakly PAC fields are non formally real and that (assuming perfectness) they satisfy the Hasse principle for the Brauer groups for all extensions of relative transcendence degree 1, and they constructed perfect weakly PAC fields which are not PAC. In particular, perfect weakly PAC fields have Property $(*)$.

The above discussion leads to the following questions in our context.

Question 4.4 *Does there exist a weakly PAC but not PAC superrosy [resp. supersimple] field?*

The positive answer would refute Conjecture 3 [resp. Conjecture 1]. The negative answer would support (but not prove) these conjectures.

Question 4.5 *Let K be an infinite perfect field with NIP satisfying Property $(*)$ [or $(*-)$]. Is it true that K is either algebraically or real closed?*

Applying the trick with adding $\sqrt{-1}$, we see that both versions of this question are equivalent. By Corollaries 8 and 9, the positive answer would imply Conjectures 2 and 4 in positive characteristic.

Question 4.6 *Does there exist a non algebraically closed, perfect weakly PAC field which [is superrosy and] has NIP?*

The positive answer would yield the negative answer to Question 4.5. The positive answer to the extended version would refute Conjecture 2. Notice that since by [5] we know that non separably closed PAC fields do not have NIP, if the answer to the above question was positive, the witness field would have to be weakly PAC but not PAC.

In any case, in order to find counter-examples to Conjectures 1, 2, 3 or 4, one could try to construct suitable non PRC fields with Property $(*-)$; possible candidates could be among perfect weakly PAC but not PAC fields. Or, try to prove the conjectures by showing that there are no such fields.

A generalization of Corollary 8 to the characteristic zero case is an open problem. Of course, Thereom 7 yields the divisibility of the value groups, but the problem is with the residue fields.

Conjecture 11 *Every non-trivial valuation on a superrosy field has divisible value group and either algebraically or real closed residue field.*

Another idea of attacking Conjecture 2 by means of Corollary 8 (or rather Conjecture 11), suggested to the author by E. Hrushovski, is to try to show that a superrosy field with NIP is the residue field with respect to some non-trivial valuation on another superrosy field with NIP, and use Corollary 8 (or Conjecture 11 in the zero characteristic case) to conclude that the original field is either algebraically or real closed.

As was mentioned in the introduction, K. Dupont has a different approach to Conjecture 4 (and so also to Conjecture 2). After adding $\sqrt{-1}$ to the field, the goal is to show that either for every finite extension L of K and for every prime number n, $(L^*)^n = L^*$ (as then K is algebraically closed by Lemma 2.4), or there is a non-trivial definable valuation on K. Assume that the first possibility fails for some L and n. Put $T = (L^*)^n$. It follows from the assumptions and Remark 2.5 that L is perfect, so $n \neq \text{char}(L)$. The first goal is to show that $\mathcal{O}_T$ is a definable valuation ring on L. Further, K. Dupont deduced from [14] that $\mathcal{O}_T$ is non-trivial if and only if the family of sets $\{aT + a : a \in L^*\}$ is a subbasis of open neighborhoods of zero of a V-topology on L. The second goal of her project is to show (using the NIP assumption) that the last condition holds.

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