EVERYWHERE MEAGRE AND EVERYWHERE NULL SETS

JAN KRASZEWSKI

Abstract. We introduce new classes of small subsets of the reals, having natural combinatorial definitions, namely everywhere meagre and everywhere null sets. We investigate properties of these sets, in particular we show that these classes are closed under taking products and projections. We also prove several relations between these classes and other well-known classes of small subsets of the reals.

1. Introduction and definitions

In 1990 Rosłanowski introduced in [15] a new σ-ideal of subsets of the Cantor space 2^ω (closely connected with Mycielski ideals – cf. [11]):

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall T \in [\omega]^\omega)\, A\restriction T \neq 2^T\},$$

which was later thoroughly investigated by many people (see e.g. [4] or [14]). As any set $A \in \mathbb{B}_2$ has the property that its section on *every* infinite set $T \subseteq \omega$ is not the whole 2^T, we can call sets from $\mathbb{B}_2$ *everywhere not everything* sets. It is natural to ask what sets we will obtain if we expect them to be everywhere smaller then just "not everything".

It is an easy observation, that a set is *everywhere countable* if and only if it is countable. However, we can modify slightly a definition of everywhere countable sets to obtain a reasonable σ-ideal $\mathcal{I}_0$. It was done by Repický in [14]:

$$\mathcal{I}_0 = \{A \subseteq 2^\omega : (\forall T \in [\omega]^\omega)(\exists S \in [T]^\omega)\, |A\restriction S| \leq \omega\}.$$

In this paper we consider another notion of smallness. Namely, we focus our attention on everywhere meagre and everywhere null sets.

Definition. A set $A \subseteq 2^\omega$ is called *everywhere meagre* (resp. *everywhere null*) if for every infinite set $T \subseteq \omega$ the set $A\restriction T = \{x\restriction T : x \in A\}$ is meagre (resp. null) in 2^T. We denote the families of everywhere meagre and everywhere null sets by $\mathcal{EM}$ and $\mathcal{EN}$, respectively.

Straight from the definitions we get $\mathcal{I}_0 \subseteq \mathcal{EM} \subseteq \mathbb{B}_2 \cap \mathcal{M}$ and $\mathcal{I}_0 \subseteq \mathcal{EN} \subseteq \mathbb{B}_2 \cap \mathcal{N}$, where $\mathcal{M}$ and $\mathcal{N}$ stand for σ-ideals of meagre sets and null sets, respectively. Further on, we will show another characterization of everywhere meagre and everywhere null sets and prove their several interesting properties. We will also investigate what relations there are between these sets and other kinds of small subsets of the Cantor space.

In our considerations we use standard set-theoretical notation and terminology from [1]. Recall that the cardinality a set X is denoted by $|X|$. The power set

2000 *Mathematics Subject Classification.* 03E05, 03E17.
Key words and phrases. Mycielski ideal, everywhere meagre sets, everywhere null sets, universally meagre sets, universally null sets.

of a set X is denoted by $\mathcal{P}(X)$. If $\varphi : X \to Y$ is a function and $A \subseteq X, B \subseteq Y$ then $\varphi[A]$ denotes the image of A and $\varphi^{-1}[B]$ denotes the pre-image of B. By $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ we denote projections on first and second coordinate, respectively. The set of all infinite subsets of ω we denote by $[\omega]^\omega$. The σ-ideal generated by closed null sets is denoted by $\mathcal{E}$.

Let INJ denote the set of all injections from ω into ω. For $\varphi \in \mathrm{INJ}$ we define a corresponding surjection $\Phi : 2^\omega \to 2^\omega$ by $\Phi(x) = x \circ \varphi$.

Let $\mathcal{J}$ be a σ-ideal of subsets of 2^ω. We say that $\mathcal{J}$ is *productive* if $\mathcal{J}$ if and only if for every $A \subseteq 2^\omega$ and $\varphi \in \mathrm{INJ}$ if $A \in \mathcal{J}$ then so is $\Phi^{-1}[A]$. We say that $\mathcal{J}$ has WFP (Weak Fubini Property) if for every $A \subseteq 2^\omega$ and $\varphi \in \mathrm{INJ}$ if $\Phi^{-1}[A]$ is in $\mathcal{J}$ then so is A (for more details – see [8]).

We can intuitively interpret these definitions in such a way that justifies their names. Namely, we can say that $\mathcal{J}$ is productive if for every $T \in [\omega]^\omega$ and every set $A \subseteq 2^T$ if A is in $\mathcal{J}$ then the cylinder $A \times 2^{\omega \setminus T}$ is in $\mathcal{J}$. Similarly, $\mathcal{J}$ has WFP if for every $T \in [\omega]^\omega$ and every $A \subseteq 2^T$ if the cylinder $A \times 2^{\omega\setminus T}$ is in $\mathcal{J}$ then its projection into 2^T, that is A, is also in $\mathcal{J}$.

Let PIF denotes the family of all partial infinite functions from ω into $\{0, 1\}$. For every $\sigma \in \mathrm{PIF}$ we put $[\sigma] = \{x \in 2^\omega : \sigma \subseteq x\}$. Let $\mathbb{S}_2$ be the σ-ideal generated by the family $\{[\sigma] : \sigma \in \mathrm{PIF}\}$. This σ-ideal was introduced and thoroughly investigated in [3]. It is well-known that $\mathbb{S}_2 \subseteq \mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$.

Straight from their definitions we obtain that σ-ideals $\mathcal{M}$, $\mathcal{N}$, $\mathcal{E}$ and $\mathbb{S}_2$ are productive and have WFP. Moreover, $\mathbb{S}_2$ is the least nontrivial productive σ-ideal of subsets of 2^ω.

For a family $\mathcal{A} \subseteq \mathcal{P}(2^\omega)$ we define the following cardinal invariants:

$$\begin{aligned}\mathrm{add}(\mathcal{A}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A} \ \&\ \bigcup \mathcal{B} \notin \mathcal{A}\},\\ \mathrm{cov}(\mathcal{A}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{A} \ \&\ \bigcup \mathcal{B} = 2^\omega\},\\ \mathrm{non}(\mathcal{A}) &= \min\{|B| : B \subseteq 2^\omega \ \&\ B \notin \mathcal{A}\}.\end{aligned}$$

Observe that if $\mathcal{A} \subseteq \mathcal{A}'$ then $\mathrm{cov}(\mathcal{A}) \geq \mathrm{cov}(\mathcal{A}')$ and $\mathrm{non}(\mathcal{A}) \leq \mathrm{non}(\mathcal{A}')$.

We will investigate relations between classes $\mathcal{EM}$ and $\mathcal{EN}$ and other classes of small subsets of the reals: strongly meagre sets $\mathcal{SM}$, strongly null sets $\mathcal{SN}$, universally meagre sets $\mathcal{UM}$ and universally null sets $\mathcal{UN}$. We will also consider Marczewski null sets s_0. For their definitions and properties we refer the reader to [1], [17] and [2].

2. Operation p

In [9] the following operation on families of subsets of 2^ω was introduced:

$$p(\mathcal{A}) = \{A \subseteq 2^\omega : (\forall \varphi \in \mathrm{INJ})\, \Phi[A] \in \mathcal{A}\}.$$

One can show that p is a topological interior operator. We will need the following basic properties of this operation.

Theorem 2.1. *Let $\mathcal{A} \subseteq \mathcal{P}(2^\omega)$. Then*

(a) $p(\mathcal{A}) \subseteq \mathcal{A}$;
(b) *if $\mathcal{A}$ is a σ-ideal, then so is $p(\mathcal{A})$;*
(c) $\mathrm{add}(p(\mathcal{A})) \geq \mathrm{add}(\mathcal{A})$;
(d) $\mathrm{cov}(p(\mathcal{A})) \geq \mathrm{cov}(\mathcal{A})$;
(e) $\mathrm{non}(p(\mathcal{A})) = \mathrm{non}(\mathcal{A})$.

Proof. We obtain (a) straight from the definition and (d) straight from (a). To get (b) we observe that $\Phi[\bigcup_{n<\omega} A_n] = \bigcup_{n<\omega} \Phi[A_n]$.

To prove (c), fix $\kappa < \mathrm{add}(\mathcal{A})$ and consider a family $\{A_\alpha : \alpha < \kappa\} \subseteq p(\mathcal{A})$. As for any $\varphi \in \mathrm{INJ}$ and $\alpha < \kappa$ we have $\Phi[A_\alpha] \in \mathcal{A}$, so $\Phi[\bigcup_{\alpha<\kappa} A_\alpha] = \bigcup_{\alpha<\kappa} \Phi[A_\alpha] \in \mathcal{A}$. Hence $\bigcup_{\alpha<\kappa} A_\alpha \in p(\mathcal{A})$, and, consequently, $\kappa < \mathrm{add}(p(\mathcal{A}))$ which ends the proof.

Finally, to prove (e) it is enough to show that $\mathrm{non}(p(\mathcal{A})) \geq \mathrm{non}(\mathcal{A})$. But if $A \notin p(\mathcal{A})$, then $\Phi[A] \notin \mathcal{A}$ for some $\varphi \in \mathrm{INJ}$. As $|\Phi[A]| \leq |A|$, we are done. □

The next theorem justifies the introduction of this operation.

Theorem 2.2. $\mathcal{EM} = p(\mathcal{M})$, $\mathcal{EN} = p(\mathcal{N})$.

Proof. To prove $\mathcal{EM} \subseteq p(\mathcal{M})$ let us fix $A \in \mathcal{EM}$ and $\varphi \in \mathrm{INJ}$. Let $T = \mathrm{range}(\varphi)$. Then the set $A \restriction T$ is meagre, so $A = \bigcup_{n<\omega} D_n$ for some nowhere dense sets $D_n \subseteq 2^T$. For every $n < \omega$ we put $D'_n = \{x \in 2^\omega : x \restriction T \in D_n\}$. Then the set $\Phi[D'_n]$ is nowhere dense in 2^ω and $\Phi[A] \subseteq \bigcup_{n<\omega} \Phi[D'_n]$. Hence the set $\Phi[A]$ is meagre and we are done.

To get the other inclusion, let us fix $A \in p(\mathcal{M})$ and $T \in [\omega]^\omega$. Let $\varphi : \omega \to T$ be any bijection. Then $\varphi \in \mathrm{INJ}$, so the set $\Phi[A]$ is meagre and we have $\Phi[A] = \bigcup_{n<\omega} D_n$ for some nowhere dense sets $D_n \subseteq 2^\omega$. As the sets $\Phi^{-1}[D_n] \restriction T$ are nowhere dense in 2^T and $A \restriction T \subseteq \bigcup_{n<\omega}(\Phi^{-1}[D_n] \restriction T)$, we obtain that $A \restriction T$ is meagre in 2^T, which ends the first part of the proof.

In a similar way we can also show that $\mathcal{EN} = p(\mathcal{N})$. □

As an immediate consequence of Theorems 2.1 and 2.2 we obtain that everywhere meagre sets and everywhere null sets form σ-ideals . Moreover, we have the following corollary.

Corollary 2.3.

$$\begin{aligned} \mathrm{add}(\mathcal{EM}) &\geq \mathrm{add}(\mathcal{M}), & \mathrm{add}(\mathcal{EN}) &\geq \mathrm{add}(\mathcal{N}), \\ \mathrm{cov}(\mathcal{EM}) &\geq \mathrm{cov}(\mathcal{M}), & \mathrm{cov}(\mathcal{EN}) &\geq \mathrm{cov}(\mathcal{N}), \\ \mathrm{non}(\mathcal{EM}) &= \mathrm{non}(\mathcal{M}), & \mathrm{non}(\mathcal{EN}) &= \mathrm{non}(\mathcal{N}). \end{aligned}$$

Remark 2.4. Following Theorem 2.2, it seems reasonably to define a family of *everywhere* $\mathcal{J}$ *sets* for any σ-ideal $\mathcal{J}$ of subsets of 2^ω as follows: $\mathcal{EJ} = p(\mathcal{J})$. For example, one can show that $\mathcal{I}_0 \subseteq \mathcal{E}\mathbb{S}_2$. As $\mathcal{A} \subseteq \mathcal{A}'$ implies $p(\mathcal{A}) \subseteq p(\mathcal{A}')$, it strengthens our observation from the introduction that $\mathcal{I}_0 \subseteq \mathcal{EM} \cap \mathcal{EN}$.

3. Products of $\mathcal{EM}$ and $\mathcal{EN}$ sets

In this section we show that the property of being everywhere meagre and everywhere null is preserved under taking products.

To begin with, we need a suitable definition.

Definition. Let $h : 2^\omega \times 2^\omega \to 2^\omega$ be a standard homeomorphism, given by conditions $h(x,y)(2n) = x(n)$, $h(x,y)(2n+1) = y(n)$ for $n \in \omega$. We say that a set $A \subseteq 2^\omega \times 2^\omega$ is *everywhere meagre* (resp. *everywhere null*) if $h[A] \subseteq 2^\omega$ is everywhere meagre (resp. everywhere null).

Let Even and Odd be the sets of even and odd natural numbers, respectively. Now we can formulate and prove the following theorem.

Theorem 3.1. *For every everywhere meagre (resp. everywhere null) sets* $A, B \subseteq 2^\omega$*, the set* $A \times B \subseteq 2^\omega \times 2^\omega$ *is everywhere meagre (resp. everywhere null).*

Proof. Let us fix sets $A, B \subseteq 2^\omega$, which are everywhere meagre (the proof for everywhere null sets goes identically) and consider the set $h[A \times B]$. Let us fix $T \in [\omega]^\omega$. If $T \subseteq \text{Even}$ then the set $h[A \times B] \restriction T$ is homeomorphic to the set $A \restriction \{\frac{n}{2} : n \in T\}$, which is meagre in $2^{\{\frac{n}{2} : n \in T\}}$. The case $T \subseteq \text{Odd}$ is analogous. Suppose now that $T \cap \text{Even} \neq \emptyset \neq T \cap \text{Odd}$. Without loss of generality we can assume that the set $T \cap \text{Even}$ is infinite. Then the set $h[A \times B] \restriction (T \cap \text{Even})$ is meagre. But this implies that the set $h[A \times B] \restriction (T \cap \text{Even}) \times h[A \times B] \restriction (T \cap \text{Odd})$, which is homeomorphic to the set $h[A \times B] \restriction T$, is also meagre. Hence $h[A \times B] \in \mathcal{EM}$, which ends the proof. □

To sum up this section, we prove that classes $\mathcal{EM}$ and $\mathcal{EN}$ are closed under projections.

Theorem 3.2. *For every everywhere meagre (resp. everywhere null) set $A \subseteq 2^\omega \times 2^\omega$ the set $\pi_1[A] \subseteq 2^\omega$ is everywhere meagre (resp. everywhere null).*

Proof. We will prove only the 'meagre' case (the 'null' case could be proved identically).

Let us fix a set $A \subseteq 2^\omega \times 2^\omega$, which is everywhere meagre. Then $h[A] \in \mathcal{EM}$. But for every $T \in [\omega]^\omega$ the set $\pi_1[A] \restriction T$ is homeomorphic to the set $h[A] \restriction \{2t : t \in T\}$, which is meagre. □

4. Relations with other small sets

In this section we prove several relations between classes $\mathcal{EM}$ and $\mathcal{EN}$ and other well-known classes of small subsets of the reals.

Repický in [14] proved that there exist perfect sets which are in $\mathcal{I}_0$. Therefore they are also in $\mathcal{EM}$ and $\mathcal{EN}$, which implies that both these σ-ideals are not included in s_0. As universally meagre and universally null sets are Marczewski null, we get that $\mathcal{EM}$ is not included in $\mathcal{UM}$ and $\mathcal{EN}$ is not included in $\mathcal{UN}$.

On the other hand, we will show that not every universally meagre set is everywhere meagre. The proof is a modification of the consideration for very meagre sets, presented in [10].

Theorem 4.1. *There exists a set $A \subseteq 2^\omega$, which is universally meagre, but not everywhere meagre.*

Proof. We will construct a set $F \subseteq 2^\omega \times 2^\omega$ such that $F \in \mathcal{UM} \setminus \mathcal{EM}$. Then the set $A = h[F]$, when $h : 2^\omega \times 2^\omega \to 2^\omega$ is the standard homeomorphism mentioned in Section 3, will be the set we are looking for.

Grzegorek proved (cf. [6],[7]) that there exists a bijection $F : S \to T$ for some $S \notin \mathcal{M}$ and $T \in \mathcal{UM}$. We treat this bijection as a subset of $2^\omega \times 2^\omega$. As $\pi_1[F] = S \notin \mathcal{M}$, then according to Theorem 3.2 we obtain that F is not everywhere meagre.

Suppose now that F is not universally meagre. Thus there exist a non-meagre subset B of a certain perfect Polish space Y and a Borel one-to-one function $f : B \to F$ (cf. [17]). But then the set B and a function $\pi_2 \circ f : B \to T$ contradict the fact that $T \in \mathcal{UM}$. Hence we obtain $F \in \mathcal{UM}$, which ends the proof. □

The same proof works for the 'null' case (using the fact from [5] on the existence of a universally null set A such that $|A| = \text{non}(\mathcal{N})$).

Theorem 4.2. *There exists a set $A \subseteq 2^\omega$, which is universally null, but not everywhere null.*

From now on, we will treat the space 2^ω as a group (identifying it with $\mathbb{Z}_2^\omega$ with the standard product group structure). To present our further results, we will need another definition, introduced in [16]. Let us recall, that a family $\mathcal{A} \subseteq \mathcal{P}(2^\omega)$ is *translation invariant* if for every $A \in \mathcal{A}$ and $x \in 2^\omega$ we have $x + A \in \mathcal{A}$.

Definition. For any translation invariant families $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(2^\omega)$ we put

$$\mathcal{G}_t(\mathcal{B}, \mathcal{A}) = \{B \subseteq 2^\omega : (\forall A \in \mathcal{A})\, A + B \in \mathcal{B}\},$$

where $A + B = \{a + b : a \in A,\, b \in B\}$.
In applications we will usually assume that $\mathcal{A} \subseteq \mathcal{B}$.

Straight from this definition we obtain the following proposition.

Proposition 4.3. *(a)* $\mathcal{A} \subseteq \mathcal{A}' \Rightarrow \mathcal{G}_t(\mathcal{B}, \mathcal{A}') \subseteq \mathcal{G}_t(\mathcal{B}, \mathcal{A})$,
(b) $\mathcal{B} \subseteq \mathcal{B}' \Rightarrow \mathcal{G}_t(\mathcal{B}, \mathcal{A}) \subseteq \mathcal{G}_t(\mathcal{B}', \mathcal{A})$.

The definition of strongly meagre sets states that $\mathcal{SM} = \mathcal{G}_t(\mathcal{P}(2^\omega) \setminus \{2^\omega\}, \mathcal{N})$ and it was proved by Galvin, Mycielski and Solovay that $\mathcal{SN} = \mathcal{G}_t(\mathcal{P}(2^\omega) \setminus \{2^\omega\}, \mathcal{M})$. In 1996 Pawlikowski proved in [13] that $\mathcal{SN} = \mathcal{G}_t(\mathcal{N}, \mathcal{E})$ and $\mathcal{SM} \subseteq \mathcal{G}_t(\mathcal{M}, \mathcal{E})$ (it is consistent that the latter inclusion is proper). In 2003 Kraszewski proved the following theorem.

Theorem 4.4 ([9])**.** *Let $\mathcal{J}$ be a translation invariant σ-ideal of subsets of 2^ω, which is productive and has WFP. Then*

$$p(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathbb{S}_2).$$

As an immediate consequence of Theorems 2.2 and 4.4, Proposition 4.3 and Pawlikowski's and Repický's results we obtain the following corollary.

Corollary 4.5. *(a)* $\mathcal{SM} \subseteq \mathcal{G}_t(\mathcal{M}, \mathcal{E}) \subsetneq \mathcal{G}_t(\mathcal{M}, \mathbb{S}_2) = \mathcal{EM}$,
(b) $\mathcal{SN} \subsetneq \mathcal{G}_t(\mathcal{N}, \mathbb{S}_2) = \mathcal{EN}$.

Remark 4.6. The fact that every strongly null set is everywhere null could be proved straightforwardly. Indeed, the σ-ideal of null subsets of 2^ω is closed under taking uniformly continuous images. Hence, for every $A \in \mathcal{SN}$ and $\varphi \in \mathrm{INJ}$ we have $\Phi[A] \in \mathcal{SN}$, which means that $A \in p(\mathcal{SN}) \subseteq p(\mathcal{N}) = \mathcal{EN}$.

In the introduction we observed that $\mathcal{EM}, \mathcal{EN} \subseteq \mathbb{B}_2$. This result can be strengthened. In order to do this, we need two lemmas.

Lemma 4.7. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \subseteq \mathcal{P}(2^\omega)$ be translation invariant families. Then*

$$\mathcal{G}_t(\mathcal{G}_t(\mathcal{C}, \mathcal{B}), \mathcal{A}) = \mathcal{G}_t(\mathcal{G}_t(\mathcal{C}, \mathcal{A}), \mathcal{B}).$$

Proof. As $C \in \mathcal{G}_t(\mathcal{G}_t(\mathcal{C}, \mathcal{B}), \mathcal{A})$ if and only if $(\forall A \in \mathcal{A})(\forall B \in \mathcal{B})\,(C + A) + B \in \mathcal{C}$ and $C \in \mathcal{G}_t(\mathcal{G}_t(\mathcal{C}, \mathcal{A}), \mathcal{B})$ if and only if $(\forall B \in \mathcal{B})(\forall A \in \mathcal{A})\,(C + B) + A \in \mathcal{C}$, we are done. □

Before we formulate the other lemma, we will simplify notation (cf. [9]). For any translation invariant family $\mathcal{A}$ of subsets of 2^ω we put $s(\mathcal{A}) = \mathcal{G}_t(\mathcal{P}(2^\omega) \setminus \{2^\omega\}, \mathcal{A})$. In [16] Seredyński observed that $\mathcal{A} \subseteq s(s(\mathcal{A}))$.

Lemma 4.8. *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(2^\omega)$ be translation invariant families. Then*

$$\mathcal{G}_t(\mathcal{B}, \mathcal{A}) \subseteq \mathcal{G}_t(s(\mathcal{A}), s(\mathcal{B})).$$

Proof. Using Proposition 4.3 and Lemma 4.7 we obtain

$$\mathcal{G}_t(\mathcal{B},\mathcal{A}) \subseteq \mathcal{G}_t(s(s(\mathcal{B})),\mathcal{A}) = \mathcal{G}_t(s(\mathcal{A}),s(\mathcal{B})).$$

□

Corollary 4.9. *If $A \in \mathcal{EM}$ and $B \in \mathcal{SN}$ then $A + B \in \mathbb{B}_2$.*

Proof. From Lemma 4.8 we have $\mathcal{EM} = \mathcal{G}_t(\mathcal{M},\mathbb{S}_2) \subseteq \mathcal{G}_t(s(\mathbb{S}_2),\mathcal{SN})$. But in [9] it is proved that $s(\mathbb{S}_2) = \mathbb{B}_2$, which ends the proof. □

In the same way we can prove the dual result.

Corollary 4.10. *If $A \in \mathcal{EN}$ and $B \in \mathcal{SM}$ then $A + B \in \mathbb{B}_2$.*

5. Problems

We have proved that $\mathcal{SM} \subseteq \mathcal{EM}$ and $\mathcal{UM} \not\subseteq \mathcal{EM}$. In [12] Nowik, Scheepers and Weiss defined AFC' sets – sets which are perfectly meagre in transitive sense. It is known that $\mathcal{SM} \subseteq AFC' \subseteq \mathcal{UM}$. So we can pose the following question:

Problem 1. *Does there exist a perfectly meagre set in transitive sense which is not everywhere meagre?*

In [14] Repický constructed (in ZFC) a set $A \in \mathbb{B}_2 \setminus \mathcal{I}_0$. The referee pointed me out that if X and Y are a Sierpiński set and a Luzin set respectively then $X \in \mathcal{EM} \setminus \mathcal{EN}$, $Y \in \mathcal{EN} \setminus \mathcal{EM}$ and $X \cup Y \in \mathbb{B}_2 \setminus (\mathcal{EM} \cup \mathcal{EN})$. Moreover, he observed that assuming Continuum Hypothesis we can construct such a scale $\{f_\alpha : \alpha < \omega_1\} \subseteq \omega^\omega$ that for a set $Z \subseteq 2^\omega$ of the characteristic functions of the ranges of the f_α we have $Z \in (\mathcal{EM} \cap \mathcal{EN}) \setminus \mathcal{I}_0$. So we can consistently differentiate these four σ-ideals . The problem is how to construct analogous examples in ZFC.

Problem 2. *Construct (in ZFC) sets $B, C \subseteq 2^\omega$ such that $B \in \mathcal{EM} \setminus \mathcal{I}_0$ and $C \in \mathbb{B}_2 \setminus \mathcal{EM}$ (and the same for everywhere null sets).*

As we could observe, in this paper there is a full symmetry between the 'meagre' case and the 'null' case. It leads us to the third problem.

Problem 3. *Find a property that differentiate the 'meagre' case from the 'null' case. Construct (in ZFC) sets $D, F \subseteq 2^\omega$ such that $D \in \mathcal{EM} \setminus \mathcal{EN}$ and $F \in \mathcal{EN} \setminus \mathcal{EM}$.*

References

[1] T. Bartoszyński, H. Judah, *Set Theory: On the structure of the real line*, A. K. Peters, Wellesley, Massachusetts 1995.
[2] J. Brendle, *Strolling through paradise*, Fund. Math. **148** (1995), 1–25.
[3] J. Cichoń, J. Kraszewski, *On some new ideals on the Cantor and Baire spaces*, Proc. Amer. Math. Soc. **126** (1998), 1549–1555.
[4] J. Cichoń, A. Rosłanowski, J. Steprāns, B. Węglorz, *Combinatorial properties of the ideal $\mathbb{B}_2$*, J. Symb. Logic **58** (1993), 42–54.
[5] E. Grzegorek, *Solution to a problem of Banach on σ-fields without continous measures*, Bull. Acad. Pol. Sci. **28** (1980), 7–10.
[6] E. Grzegorek, *Always of the first category sets*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **6** (1984), 139–147.
[7] E. Grzegorek, *Always of the first category sets II*, Rend. Circ. Mat. Palermo, II. Ser. Suppl. **10** (1985), 43–48.
[8] J. Kraszewski, *Properties of ideals on generalized Cantor spaces*, J. Symb. Logic **66** (2001), 1303–1320.

[9] J. Kraszewski, *Transitive properties of the ideal* $\mathbb{S}_2$, Real Anal. Ex. **29** (2003/2004), 629–638.
[10] M. Kysiak, *On Erdős-Sierpiński duality for Lebesgue measure and Baire category*, Master's thesis, Warsaw 2000 (in Polish).
[11] J. Mycielski, *Some new ideals of sets on the real line*, Colloq. Math. **20** (1969), 71–76.
[12] A. Nowik, M. Scheepers, T. Weiss, *The algebraic sum of sets of real numbers with strong-measure zero sets*, J. Symb. Logic **63** (1998), 301–324.
[13] J. Pawlikowski, *A characterization of strong measure zero sets*, Isr. J. Math. **93** (1996), 171–183.
[14] M. Repický, *Mycielski ideal and the perfect set theorem*, Proc. Amer. Math. Soc. **132** (2004), 2141–2150.
[15] A. Rosłanowski, *On game ideals*, Colloq. Math. **59** (1990), 159–168.
[16] W. Seredyński, *Some operations related with translation*, Colloq. Math. **57** (1989), 203–219.
[17] P. Zakrzewski, *Universally meager sets*, Proc. Amer. Math. Soc. **129** (2001), 1793–1798.

Mathematical Institute, Wrocław University, pl. Grunwaldzki 2/4, 50-156 Wrocław, Poland

E-mail address: kraszew@math.uni.wroc.pl