PRODUCTIVITY VERSUS WEAK FUBINI PROPERTY

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Abstract. We construct a σ-ideal of subsets of the Cantor space which is productive but does not have the Weak Fubini Property. In the construction we use a combinatorial lemma which is of its own interest.

1. Introduction

In 2001 Kraszewski in [4] defined a class of *productive* σ-ideals of subsets of the Cantor space 2^ω and observed that both σ-ideals of meagre sets $\mathcal{M}$ and of null sets $\mathcal{N}$ are in this class. Next, from every productive σ-ideal $\mathcal{J}$ one can produce a σ-ideal $\mathcal{J}_\kappa$ of subsets of the generalized Cantor space 2^κ. In particular, starting from meagre sets and null sets in 2^ω we obtain meagre sets and null sets in 2^κ, respectively. This description gives us a powerful tool for investigating combinatorial properties of ideals on 2^κ, which was done in [4].

However, some theorems needed an additional assumption that a σ-ideal of subsets of 2^ω we started from had the Weak Fubini Property. The natural question posed in [4] was whether this extra assumption could be omitted.

The weaker version of this question is whether every productive σ-ideal of subsets of 2^ω has the Weak Fubini Property. In this paper we answer it negatively. In the construction we make use of an interesting combinatorial lemma on existence of some special family of subsets of ω.

2. Preliminaries

In this paper we deal with the Cantor space 2^ω interpreted as the set of all functions from ω into the set $\{0,1\}$, endowed with the standard product topology.

We use standard set-theoretical notation from [1] and [3]. In particular, if X is a set and κ is a cardinal then $[X]^\kappa$ and $[X]^{<\kappa}$ stand for the families of subsets of X of cardinality κ and smaller then κ, respectively. The set of natural numbers is denoted by ω. As we identify this set with the cardinal number associated with countable infinite sets, we denote this cardinal with the same symbol. The cardinal number continuum, i.e. the cardinality of $\mathcal{P}(\omega)$, is denoted by $\mathfrak{c}$. We identify $\mathcal{P}(\omega)$ with 2^ω in the natural way using characteristic functions. We also very often identify 2^ω with $2^T \times 2^{\omega\setminus T}$ for $T \subseteq \omega$. If $f : X \to Y$ is a function, $A \subseteq X$ and $B \subseteq Y$, then $f[A]$ and $f^{-1}[B]$ denote the image of A and pre-image of B, respectively.

For an ideal $\mathcal{J}$ of subsets of a space X, by $\mathcal{J}^*$ we denote its dual filter, i.e. $\mathcal{J}^* = \{X \setminus A : A \in \mathcal{J}\}$.

For a Polish space X, we denote by $K(X)$ the space of all compact subsets of X with Vietoris topology (see [3]). Recall that for a Polish space X the space $K(X)$ is also Polish and the space $K(2^\omega)$ is homeomorphic to 2^ω.

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Let INJ denote the set of all injections from ω into ω and let CIR denote the set of all injections from ω to ω with co-infinite range. For $\varphi \in$ INJ we define a corresponding surjection $\Phi : 2^\omega \to 2^\omega$ by $\Phi(x) = x \circ \varphi$ (therefore Φ_i corresponds to φ_i, Ψ to ψ etc.).

Let $\mathcal{A}$ be a family of subsets of 2^ω. We define

$$\omega(\mathcal{A}) = \{A \subseteq 2^\omega : (\exists \varphi \in \mathrm{INJ})\, \Phi[A] \in \mathcal{A}\}.$$

If $A \in \omega(\mathcal{A})$ then any $\varphi \in$ INJ such that $\Phi[A] \in \mathcal{A}$ is called *a witness* for A. We have always $\mathcal{A} \subseteq \omega(\mathcal{A})$ as an identity is a witness. By $\sigma(\mathcal{A})$ we denote the σ-ideal generated by $\mathcal{A}$.

Let $\mathcal{J}$ be a σ-ideal of subsets of 2^ω. We say that $\mathcal{J}$ is *productive* if $\omega(\mathcal{J}) \subseteq \mathcal{J}$. One can show that $\mathcal{J}$ is productive if and only if for every $A \subseteq 2^\omega$ and $\varphi \in$ INJ if $A \in \mathcal{J}$ then so is $\Phi^{-1}[A]$. We say that $\mathcal{J}$ has WFP (Weak Fubini Property) if for every $A \subseteq 2^\omega$ and $\varphi \in$ INJ if $\Phi^{-1}[A]$ is in $\mathcal{J}$ then so is A.

We can intuitively interpret these definitions in such a way that justifies their names. Namely, we can say that $\mathcal{J}$ is productive if for every $T \in [\omega]^\omega$ and every set $A \subseteq 2^T$ if A is in $\mathcal{J}$ then the cylinder $A \times 2^{\omega\setminus T}$ is in $\mathcal{J}$. Similarly, $\mathcal{J}$ has WFP if for every $T \in [\omega]^\omega$ and every $A \subseteq 2^T$ if the cylinder $A \times 2^{\omega\setminus T}$ is in $\mathcal{J}$ then its projection into 2^T, that is A, is also in $\mathcal{J}$.

Straight from their definitions we obtain that σ-ideals of meagre and null sets are productive and have WFP, as well as the σ-ideal generated by closed null sets.

The following fact shows that every σ-ideal has its productive closure.

Fact 2.1. *For any σ-ideal $\mathcal{J}$ of subsets of 2^ω the σ-ideal $\sigma(\omega(\mathcal{J}))$ is productive.*

Proof. It is enough to prove that $\omega(\sigma(\omega(\mathcal{J}))) \subseteq \sigma(\omega(\mathcal{J}))$. Let $A \subseteq 2^\omega$ be a member of $\omega(\sigma(\omega(\mathcal{J})))$ and let $\varphi \in$ INJ be its witness. It means that $\Phi[A] \in \sigma(\omega(\mathcal{J}))$ which implies $\Phi[A] \subseteq \bigcup_{i<\omega} A_i$ for some family $\{A_i : i < \omega\} \subseteq \omega(\mathcal{J})$. Thus

$$A \subseteq \Phi^{-1}[\Phi[A]] \subseteq \bigcup_{i<\omega} \Phi^{-1}[A_i].$$

Let $\varphi_i \in$ INJ be a witness for A_i. For each $i < \omega$ we have $(\Phi_i \circ \Phi)[\Phi^{-1}[A_i]] = \Phi_i[A_i] \in \mathcal{J}$, so $\Phi^{-1}[A_i]$ is a member of $\omega(\mathcal{J})$ and $\varphi \circ \varphi_i$ is its witness. Consequently, $\bigcup_{i<\omega} \Phi^{-1}[A_i]$ is a member of $\sigma(\omega(\mathcal{J}))$ and so is A. $\square$

Let PIF denotes the family of all partial infinite functions from ω into $\{0, 1\}$. For every $\sigma \in$ PIF we put $[\sigma] = \{x \in 2^\omega : \sigma \subseteq x\}$. Let $\mathbb{S}_2$ be the σ-ideal generated by the family $\{[\sigma] : \sigma \in \mathrm{PIF}\}$. This σ-ideal was introduced and thoroughly investigated in [2]. The more important property for us is that $\mathbb{S}_2$ is the least nontrivial productive σ-ideal of subsets of 2^ω (as $\mathbb{S}_2 = \sigma(\omega([2^\omega]^{\leq\omega})))$. Obviously, $\mathbb{S}_2$ has WFP as well.

3. Main lemma

In this section we prove a combinatorial lemma which will be essential in our construction.

We begin with an observation concerning the hyperspace of compact subsets of the Cantor space which we will be useful in the proof of the main lemma. This lemma was communicated to us by Paweł Milewski, but he claims that it is a part of topological folklore. We would like to thank him for permission to include the proof in our paper.

Lemma 3.1. *Let X, Y be perfect compact Polish spaces. Then the family of all perfect sets $P \subseteq X \times Y$ which are graphs of partial 1–1 functions from X to Y is co-meager in $K(X \times Y)$.*

Proof. Consider the space $K(X \times Y) \times X \times Y$. It is known (see [3]) that the membership relation is a closed subset of this space, i.e. the set $\{\langle P, x, y\rangle : \langle x, y\rangle \in P\}$ is closed in $K(X \times Y) \times X \times Y$. Also it is easy to check that the family of all perfect sets is co-meager in $K(X \times Y)$. Thus we may restrict our scope to perfect sets only.

We need the following fact

Lemma 3.2. *Suppose that $F \subseteq X \times Y$ is closed. Then the set $\{x \in X : |F_x| > 1\}$ is F_σ.*

Proof. Observe that $|F_x| > 1$ if and only if there exist disjoint basic open sets U, V in Y such that $F_x \cap U \neq \emptyset \neq F_x \cap V$. Thus

$$\{x \in X : |F_x| > 1\} = \bigcup_{U \cap V = \emptyset} \mathrm{proj}_X[F \cap (X \times U)] \cap \mathrm{proj}_X[F \cap (X \times V)].$$

As $X \times Y$ is compact, this set is F_σ. □

Applying this fact to $F = \{\langle P, x, y\rangle : \langle x, y\rangle \in P\}$ we get that the set

$$\{\langle P, x\rangle : \exists y_1, y_2 \in Y \;\; y_1 \neq y_2 \wedge \langle x, y_1\rangle \in P \wedge \langle x, y_2\rangle \in P\}$$

is F_σ. Projecting this set on $K(X \times Y)$ we get that the set

$$\{P \in K(X \times Y) : \exists x \;\; \exists y_1, y_2 \;\; y_1 \neq y_2 \wedge \langle x, y_1\rangle \in P \wedge \langle x, y_2\rangle \in P\}$$

is F_σ.

It follows that the family of all compact sets which are graphs of partial functions from X and Y is G_δ. To check that it is co-meager it is sufficient to check that it is dense, which is left to the reader. □

The following fact is well known.

Lemma 3.3. *For every $G \in \mathcal{M}^*$ the set $\{P \in K(2^\omega) : P \subseteq G\}$ is co-meager in $K(2^\omega)$.* □

For an infinite and co-infinite set $T \subseteq 2^\omega$ and $W \in [2^T]^{<\mathfrak{c}}$ let

$$[W] = \{x \in 2^\omega : x \restriction T \in W\}.$$

Observe that if we identify 2^ω with $2^T \times 2^{\omega \setminus T}$ then we can think of $[W]$ as $W \times 2^{\omega\setminus T}$. Let $\mathcal{W}_{<\mathfrak{c}}$ be the σ-ideal generated by all sets of the form $[W]$, for all possible W as above. One can easily check that in fact $\mathcal{W}_{<\mathfrak{c}} = \sigma(\omega([2^\omega]^{<\mathfrak{c}}))$.

Lemma 3.4. *For every set $G \in \mathcal{M}^*$ and $W \in \mathcal{W}_{<\mathfrak{c}}$ there exists a perfect set $P \subseteq G \setminus W$.*

Proof. Let $G \in \mathcal{M}^*$ and $W \in \mathcal{W}_{<\mathfrak{c}}$ be given. Then $W = \bigcup_{n\in\omega}[W_n]$, where $W_n \in [2^{T_n}]^{<\mathfrak{c}}$. We will construct a Cantor scheme $\langle P_s : s \in 2^{<\omega}\rangle$ consisting of nonempty perfect sets, with $P_{\langle\rangle} \subseteq G$, such that $P_s \cap [W_{|s|}] = \emptyset$ and put $P = \bigcap_n \bigcup_{|s|=n} P_s$.

Using Lemmas 3.1 and 3.3 we find a perfect set $Q \subseteq G$ such that for every n (under the natural identification) $Q \subseteq 2^{T_n} \times 2^{\omega\setminus T_n}$ is the graph of a partial function. Observe that $|Q \cap [W_0]| < \mathfrak{c}$, so we can find a perfect set $P_{\langle\rangle} \subseteq Q$ disjoint with $[W_0]$.

Now suppose that P_s have been constructed for $|s| \leq n$. For every P_s such that $|s| = n$ we find its two disjoint perfect subsets $Q_{s^\frown 0}$ and $Q_{s^\frown 1}$. As $Q_{s^\frown i}$ (for $i \in \{0,1\}$) is a subset of Q, it is the graph of a partial function from $2^{T_{n+1}}$ to $2^{\omega \setminus T_{n+1}}$. Thus $|Q_{s^\frown i} \cap [W_{n+1}]| < \mathfrak{c}$ for $i \in \{0,1\}$ so we may find perfect sets $P_{s^\frown i} \subseteq Q_{s^\frown i}$ $(i \in \{0,1\})$ disjoint with $[W_{n+1}]$. □

The following lemma is the main result of this section which will be used in the construction of our ideal.

Lemma 3.5. *There exists a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that $\emptyset, \omega \in \mathcal{A}$ and for every $\mathcal{F} \in [\mathrm{CIR}]^\omega$*

$$\exists A \in \mathcal{A} \ \forall f \in \mathcal{F} \ \ f^{-1}[A] \notin \mathcal{A}.$$

Proof. Let $\{\mathcal{F}_\alpha : 1 < \alpha < \mathfrak{c}\}$ be an enumeration of $[\mathrm{CIR}]^\omega$. We will inductively construct a family $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ which satisfies the following conditions for $\alpha > 1$

(1) $f^{-1}[A_\alpha] \notin \{A_\xi : \xi < \alpha\}$ for $f \in \mathcal{F}_\alpha$,
(2) $f^{-1}[A_\alpha] \neq A_\alpha$ for $f \in \mathcal{F}_\alpha$,
(3) $A_\alpha \neq f^{-1}[A_\xi]$ for $\xi < \alpha$ and $f \in \mathcal{F}_\xi$.

Observe that A_α (for $\alpha > 1$) witnesses that $\mathcal{F}_\alpha$ does not contradict the desired property of $\mathcal{A}$, i.e. we have $\forall f \in \mathcal{F}_\alpha \ \ f^{-1}[A_\alpha] \notin \mathcal{A}$. More precisely, condition (1) guarantees that $f^{-1}[A_\alpha]$ is not in the part of $\mathcal{A}$ constructed so far, condition (2) guarantees that it is not added in the step α and condition (3) that it will not be added in the following steps of our construction.

To guarantee that $\emptyset, \omega \in \mathcal{A}$ let us define $A_0 = \emptyset$ and $A_1 = \omega$. Now, suppose that A_ξ have been already constructed for $\xi < \alpha$. First we check that there are a lot of sets which satisfy condition (2).

We will show that for a fixed $f \in \mathrm{CIR}$ the set $\{A \in 2^\omega : f^{-1}[A] = A\}$ is meager in 2^ω. Indeed, observe that $f^{-1}[A] = A$ if, and only if, $\forall n \in \omega \ \ n \in A \Leftrightarrow f(n) \in A$. For $f \in \omega^\omega$ and $n \in \omega$ define $\mathrm{orb}_f(n) = \{n, f(n), f(f(n)), \ldots\}$. It follows that $f[A] = A$ if, and only if, for every $n \in \omega$ either $\mathrm{orb}_f(n) \subseteq A$ or $\mathrm{orb}_f(n) \cap A = \emptyset$, in other words χ_A is constant on every set $\mathrm{orb}_f(n)$.

Observe that for $f \in \mathrm{CIR}$ there exists $n \in \omega$ such that $\mathrm{orb}_f(n)$ is infinite, otherwise f would be onto ω. But the set of all A which are constant on a given infinite set is meager in 2^ω, which ends the proof of the claim.

As we have only countably many functions in $\mathcal{F}_\alpha$ we see that the set G of those A which satisfy condition (2) is co-meager.

Now we check that the set of those A which do not satisfy condition (1) is in $\mathcal{W}_{<\mathfrak{c}}$. Indeed, $f^{-1}[A] = A_\xi$ if, and only if, $A \cap \mathrm{rg}(f) = f[A_\xi]$. But this means that $A \in [X_f]$ for $X_f = \{f[A_\xi] : \xi < \alpha\}$. So the set of all "inappropriate" A is covered by $\bigcup_{f \in \mathcal{F}_\alpha} [X_f] \in \mathcal{W}_{<\mathfrak{c}}$.

Lemma 3.4 gives us a perfect set of A satisfying conditions (1) and (2). Finally, observe that condition (3) states that A_α must be different from $< \mathfrak{c}$ points, so obviously we may find such a set A_α in P. This A_α satisfies conditions (1)–(3). □

In the proof we used some specific relationship between the ideals $\mathcal{M}$ and $\mathcal{W}_{<\mathfrak{c}}$, namely Lemma 3.4. It is not clear whether an analogous assertion holds, if replace the ideal $\mathcal{M}$ with the ideal $\mathcal{N}$ of null subsets of the Cantor space.

Question 1. *Is is true that for every $G \in \mathcal{N}^*$ we have $G \notin \mathcal{W}_{<\mathfrak{c}}$?*

Question 2. *Is it true that for every $G \in \mathcal{N}^*$ and $W \in \mathcal{W}_{<\mathfrak{c}}$ there exists a perfect set $P \subseteq G \setminus W$?*

Clearly, the answer to both questions is positive under some additional set-theoretic assumptions, e.g. if every set of cardinality smaller than $\mathfrak{c}$ is null.

4. Construction

Now we are ready to prove that productivity does not imply the Weak Fubini Property.

Theorem 4.1. *There exists a productive σ-ideal of subsets of the Cantor space 2^ω that does not have WFP.*

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(\omega)$ be a family from Lemma 3.5. Let

$$C = \{\chi_A : A \in \mathcal{A}\} \subseteq 2^\omega.$$

Let $\varphi_0 : \omega \to \omega,\ \varphi_0(n) = 2n$. Obviously, $\varphi_0 \in \mathrm{CIR}$. We define

$$\mathcal{J} = \sigma(\omega(\sigma(\mathbb{S}_2 \cup \{\Phi_0^{-1}[C]\}))).$$

We prove that $\mathcal{J}$ is a σ-ideal we are looking for. From Fact 2.1 we obtain that $\mathcal{J}$ is productive as a productive closure of the σ-ideal $\sigma(\mathbb{S}_2 \cup \{\Phi_0^{-1}[C]\})$. So it is enough to show that $C \notin \mathcal{J}$ because then $\Phi_0^{-1}[C] \in \mathcal{J}$ is a counterexample for having WFP.

Suppose otherwise that $C \in \mathcal{J}$. Then $C \subseteq \bigcup_{i<\omega} D_i$ for some $D_i \in \omega(\sigma(\mathbb{S}_2 \cup \{\Phi_0^{-1}[C]\}))$. But then for every $i < \omega$ there exist $\psi_i \in \mathrm{INJ}$ and $S_i \in \mathbb{S}_2$ such that $D_i \subseteq \Psi_i^{-1}[S_i \cup \Phi_0^{-1}[C]]$. Hence we get

$$C \subseteq \bigcup_{i<\omega} \Psi_i^{-1}[S_i] \cup \bigcup_{i<\omega} \Psi_i^{-1}[\Phi_0^{-1}[C]].$$

But it is easy to check that $\bigcup_{i<\omega} \Psi_i^{-1}[S_i] \in \mathbb{S}_2$, so finally

$$C \subseteq \bigcup_{j<\omega} [\sigma_j] \cup \bigcup_{i<\omega} \overline{\Psi}_i^{-1}[C],$$

where $\sigma_j \in \mathrm{PIF}$ and $\overline{\psi}_i = \psi_i \circ \varphi_0 \in \mathrm{CIR}$. We can assume (shrinking the domain of σ_j if necessary) that $\mathrm{dom}(\sigma_j)$ is coinfinite and σ_j is constantly equal to 0 or constantly equal to 1. For every $i < \omega$ we fix any bijection $f_j : \omega \to \mathrm{dom}(\sigma_j)$.

Let us consider the family $\mathcal{F} = \{\overline{\psi}_i : i < \omega\} \cup \{f_j : j < \omega\} \subseteq \mathrm{CIR}$. Let $A \in \mathcal{A}$ be such that for every $x \in \mathcal{F}$ we have $x^{-1}[A] \notin \mathcal{A}$.

If $\chi_A \in \overline{\Psi}_i^{-1}[C]$ for some $i < \omega$ then $\chi_A \circ \overline{\psi}_i \in C$ so $(\exists B \in \mathcal{A})\, \chi_A \circ \overline{\psi}_i = \chi_B$. Thus $B = \overline{\psi}_i^{-1}[A]$ which is impossible.

Similarly, if $\chi_A \in [\sigma_j]$ for some $j < \omega$ then $\chi_A \circ f_j = \chi_\emptyset$ or $\chi_A \circ f_j = \chi_\omega$. But $\emptyset, \omega \in \mathcal{A}$, a contradiction.

Hence we get $\chi_A \notin C$, which ends the proof. $\square$

We have showed that every family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ having the property mentioned in Lemma 3.5 gives us a productive σ-ideal without WFP. One can prove that the converse is also true, i.e. from every productive σ-ideal without WFP we can obtain a family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ such that for every $\mathcal{F} \in [\mathrm{CIR}]^\omega$ there exists $A \in \mathcal{A}$ such that $f^{-1}[A] \notin \mathcal{A}$ for any $f \in \mathcal{F}$.

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