Transitive properties of the ideal $\mathbb{S}_2$

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Abstract

In this paper we compute transitive cardinal coefficients of the σ-ideal $\mathbb{S}_2$, the least nontrivial productive σ-ideal of subsets of the Cantor space 2^ω. We also apply transitive operations to $\mathbb{S}_2$. In particular, we show that σ-ideal of strongly $\mathbb{S}_2$ sets is equal to $\mathbb{B}_2$, one of Mycielski ideals.

0. Introduction. In this paper we investigate transitive properties of the σ-ideal $\mathbb{S}_2$. This ideal appeared for the first time in [10], but only incidentally. It was thoroughly investigated by Cichoń and Kraszewski in [5]. It turned out that cardinal characteristics of $\mathbb{S}_2$ are strongly connected with some intensively studied combinatorial properties of subsets of natural numbers (the splitting and reaping numbers). Namely,

$$\mathrm{add}(\mathbb{S}_2) = \omega_1, \quad \mathrm{cov}(\mathbb{S}_2) = \mathfrak{r}, \quad \mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}, \quad \mathrm{cof}(\mathbb{S}_2) = \mathfrak{c}.$$

Moreover, $\mathbb{S}_2$ is the least nontrivial productive σ-ideal of subsets of the Cantor space 2^ω. The notion of productivity is a powerful tool for investigating properties of ideals on generalized Cantor spaces 2^κ. For more details see [9].

In the first part of this paper we completely describe all well-known transitive cardinal characteristics of $\mathbb{S}_2$. In the second part we apply transitive operations to $\mathbb{S}_2$. In particular, we show that the σ-ideal of strongly $\mathbb{S}_2$ sets is exactly $\mathbb{B}_2$, one of Mycielski ideals.

1. Definitions and basic properties. We use standard set-theoretical notation and terminology from [2]. Recall that the cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. The power set of a set X is denoted by $\mathcal{P}(X)$. If κ is a cardinal number then $[X]^\kappa$ denotes the family of all subsets

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of the set X of cardinality κ. If $\varphi : X \to Y$ is a function then $\mathrm{rng}(\varphi)$ denotes the range of φ. If $A \subseteq Y$ then $\varphi^{-1}[A]$ denotes the pre-image of A.

Let $\mathcal{J}$ be an ideal of subsets of an abelian group G. We say that $\mathcal{J}$ is translation invariant if $A + g = \{x + g : x \in A\} \in \mathcal{J}$ for each $A \in \mathcal{J}$ and $g \in G$ and that $\mathcal{J}$ is symmetric if $-A = \{-x : x \in A\} \in \mathcal{J}$ for each $A \in \mathcal{J}$.

For an ideal $\mathcal{J}$ we consider the following cardinal numbers

$$\begin{aligned}
\mathrm{add}_t(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \wedge \neg(\exists B \in \mathcal{J})(\forall A \in \mathcal{A})(\exists g \in G)\, A \subseteq B + g\},\\
\mathrm{add}_t^*(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T \notin \mathcal{J}\},\\
\mathrm{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T = G\},\\
\mathrm{cof}_t(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \wedge \mathcal{B} \text{ is a transitive base of } \mathcal{J}\},
\end{aligned}$$

where a family $\mathcal{B} \subseteq \mathcal{J}$ is called a *transitive base* if for each $A \in \mathcal{J}$ there exists $B \in \mathcal{B}$ and $g \in G$ such that $A \subseteq B+g$. The first two of these are both called *transitive additivity* and the last two are called *transitive covering number* and *transitive cofinality*, respectively. Let us notice that all definitions of cardinal coefficients mentioned above are valid also for an arbitrary family $\mathcal{A} \subseteq \mathcal{P}(G)$.

Let us also recall that by the *uniformity* of $\mathcal{J}$ we mean the following cardinal number

$$\mathrm{non}(\mathcal{J}) = \min\{|A| : A \subseteq G \wedge A \notin \mathcal{J}\}.$$

We say that an ideal $\mathcal{J}$ is *κ-translatable* if

$$(\forall A \in \mathcal{J})(\exists B \in \mathcal{J})(\forall T \in [G]^\kappa)(\exists g \in G)\, A + T \subseteq B + g.$$

We define a *translatability number* of $\mathcal{J}$ as follows

$$\tau(\mathcal{J}) = \min\{\kappa : \mathcal{J} \text{ is not } \kappa\text{-translatable}\}.$$

In this paper we deal with the Cantor space 2^ω interpreted as the set of all functions from ω into the set $\{0, 1\}$. This space is endowed with the standard product topology. Moreover, we consider the standard product group structure on 2^ω.

We define

$$Pif = \{f : t \text{ is a function } \wedge \mathrm{dom}(f) \in [\omega]^\omega \wedge \mathrm{rng}(f) \subseteq 2\}.$$

If $f \in Pif$ then we put

$$[f] = \{x \in 2^\omega : f \subseteq x\}.$$

Let $\mathbb{S}_2$ denote the σ-ideal of subsets of 2^ω, which is generated by the family $\{[f] : f \in Pif\}$. We recall some properties of $\mathbb{S}_2$, which were proved in [5].

Fact 1.1 *(a) $\mathbb{S}_2$ is a proper σ-ideal, containing singletons, with a base consisting of Borel sets. Every $A \in \mathbb{S}_2$ is both meagre and null.*
(b) $\mathbb{S}_2$ is translation invariant.

(c) There exists a family of size $\mathfrak{c}$ of pairwise disjoint Borel subsets of 2^ω that do not belong to $\mathbb{S}_2$. $\square$

We call a family $\mathcal{F} \subseteq Pif$ *normal* if for each two different $f_1, f_2 \in \mathcal{F}$ we have $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) = \emptyset$. Directly from the definition of $\mathbb{S}_2$ we can deduce that

$$A \in \mathbb{S}_2 \iff A \subseteq \bigcup_{f \in \mathcal{F}} [f],$$

for some countable normal family $\mathcal{F} \subseteq Pif$. In [5] the following useful lemma was proved.

Lemma 1.2 *Suppose that $\{f_i : i \in I\}$ is a normal family of functions from Pif, $f \in Pif$ and $[f] \subseteq \bigcup_{i \in I}[f_i]$. Then $[f] \subseteq [f_i]$ for some $i \in I$.* $\square$

Let A, S be two infinite subsets of ω. We say that S *splits* A if $|A \cap S| = |A \setminus S| = \omega$. Let us recall a cardinal number related with a notion of splitting, introduced by Malychin in [13], namely

$$\aleph_0\text{-}\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \wedge (\forall \mathcal{A} \in [[\omega]^\omega]^\omega)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S \text{ splits } A)\}.$$

More about cardinal numbers connected with the relation of splitting can be found in [8]. It was proved in [5] that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$.

We will need one more σ-ideal. Let us define

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall X \in [\omega]^\omega)\, A \restriction X \neq 2^X\},$$

where $A \restriction X = \{x \restriction X : x \in A\}$. This is one of the Mycielski ideals and was intensively studied by many authors (cf. [7], [15], [17]).

2. Transitive cardinal coefficients of $\mathbb{S}_2$. Let $\mathcal{J}$ be an ideal of subsets of an abelian group G. The first cardinal coefficient on the stage was the transitive covering number of $\mathcal{J}$ that appeared implicitly in 1938 in the famous Rothberger theorem, which was originally formulated for classical ideals of meagre and null subsets of the real line (cf. [18]). In his general version it says that if $\mathcal{J}$ and $\mathcal{I}$ are translation invariant ideals of subsets of G, orthogonal to each other (that is there exist $A \in \mathcal{J}$ and $B \in \mathcal{I}$ such that $A \cup B = G$) then $\mathrm{cov}_t(\mathcal{J}) \leq \mathrm{non}(\mathcal{I})$. It is worth observing that the transitive covering number may be different from the covering number and $\mathbb{S}_2$ is an example.

Theorem 2.1 $\mathrm{cov}_t(\mathbb{S}_2) = \mathfrak{c}$

Proof. It is obvious that $\mathrm{cov}_t(\mathbb{S}_2) \leq \mathfrak{c}$, so it is enough to show the other inequality. Let $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. Without loss of generality we can assume that $A = \bigcup_{i<\omega}[f_i]$, where the family $\{f_i : i < \omega\} \subseteq Pif$ is normal. If $|T| < \mathfrak{c}$ then for every $i < \omega$ there exist a function $g_i : \mathrm{dom}(f_i) \to 2$ which is different from every function $f_i + t \restriction \mathrm{dom}(f_i)$, where

$t \in T$. Because the family $\{f_i : i < \omega\}$ is normal then there exists a function $x \in 2^\omega$ such that $\bigcup_{i<\omega} g_i \subseteq x$ and we have $x \notin (A + T)$ which ends the proof. $\square$

In 1985 Pawlikowski in [16] introduced the transitive cofinality and gave the complete description of transitive cofinalities of ideals of meagre and null subsets of the real line. He also mentioned a dual coefficient to the transitive cofinality. Following the way of describing cardinal characteristics of the continuum presented by Blass in [3] we will call it a transitive additivity and denote by $\mathrm{add}_t(\mathcal{J})$. Unfortunately, Pawlikowski (and then Bartoszyński and Judah in [2]) used this name and notation for yet another coefficient, introduced in [16]. In order not to make a mess we will call the latter coefficient the starred transitive additivity and denote it by $\mathrm{add}_t^*(\mathcal{J})$.

Now we calculate these coefficients for $\mathbb{S}_2$. To begin with, we observe the following general property concerning starred transitive additivity.

Proposition 2.2 *Let $\mathcal{J}$ be a proper and translation invariant σ-ideal of subsets of a group G containing all singletons. Then* $\mathrm{add}_t^*(\mathcal{J}) \leq \mathrm{non}(\mathcal{J})$.

Proof. To prove that $\mathrm{add}_t^*(\mathcal{J}) \leq \mathrm{non}(\mathcal{J})$ it is enough to observe that for every set $T \subseteq G$ such that $T \notin \mathcal{J}$ we have $|T| \geq \mathrm{add}_t^*(\mathcal{J})$ because $\{0\} + T = T \notin \mathcal{J}$ and, of course, $\{0\} \in \mathcal{J}$. $\square$

Theorem 2.3 $\mathrm{add}_t^*(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$.

Proof. As $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ then thanks to Proposition 2.2 it is enough to show that $\mathrm{add}_t^*(\mathbb{S}_2) \geq \mathrm{non}(\mathbb{S}_2)$.

Suppose now that $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. To finish the proof we show that if $|T| < \mathrm{non}(\mathbb{S}_2)$ then $A + T \in \mathbb{S}_2$. As in the proof of Theorem 2.1 we can assume that $A = \bigcup_{i<\omega}[f_i]$, where $f_i \in Pif$ form a normal family. Thus

$$A + T = \bigcup_{t \in T} A + t = \bigcup_{t \in T} \bigcup_{i<\omega} ([f_i] + t) = \bigcup_{i<\omega} \bigcup_{t \in T} [f_i + t \restriction \mathrm{dom}(f_i)]$$

Fix $i < \omega$. Let $\iota : \mathrm{dom}(f_i) \to \omega$ be a bijection. It induces a bijection $\hat{\iota} : 2^{\mathrm{dom}(f_i)} \to 2^\omega$. The image of the set $\{f_i + t \restriction \mathrm{dom}(f_i) : t \in T\} \subseteq 2^{\mathrm{dom}(f_i)}$ by $\hat{\iota}$ has cardinality strictly smaller than $\mathrm{non}(\mathbb{S}_2)$. Consequently, it can be covered by a set $\bigcup_{j<\omega}[g_j]$, for some $\{g_j : j < \omega\} \subseteq Pif$. Hence

$$\bigcup_{t \in T} [f_i + t \restriction \mathrm{dom}(f_i)] \subseteq \bigcup_{j<\omega} [\hat{\iota}^{-1}(g_j)] \in \mathbb{S}_2,$$

which ends the proof. $\square$

In order to prove results about $\mathrm{add}_t(\mathbb{S}_2)$ and $\mathrm{cof}_t(\mathbb{S}_2)$ we introduce some extra notation. For a set $X \in [\omega]^\omega$ let $(X)^\omega_\omega$ denote the family of all infinite partitions of X into infinite parts. For $P_1, P_2 \in (\omega)^\omega_\omega$ we put $P_1 \preceq P_2$ if for every $p_1 \in P_1$ there exists $p_2 \in P_2$ such that $p_2 \subseteq p_1$ (we say that P_2 *dominates* P_1). It is not difficult to observe that

$\preceq$ is a partial ordering on $(\omega)^\omega_\omega$. Let us notice that if we consider $\preceq$ on the family (ω) of all partitions of ω (which is more common) then $\{\omega\}$ (one-element partition) is the smallest element of this ordering while the partition into singletons is the greatest one. Properties of relations on partitions of ω have been intensively studied lately by Matet, Majcher-Iwanow and others; for more information cf. [14], [6] or [12].

We define an unboundedness and dominating numbers $\mathfrak{b}_\preceq$ and $\mathfrak{d}_\preceq$ in a standard way.

$$\mathfrak{b}_\preceq = \min\{|\mathcal{R}| : R \subseteq (\omega)^\omega_\omega \wedge (\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R}) R \not\preceq P\},$$
$$\mathfrak{d}_\preceq = \min\{|\mathcal{R}| : R \subseteq (\omega)^\omega_\omega \wedge (\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R}) P \preceq R\}.$$

We have the following well-known lemma.

Lemma 2.4 $\mathfrak{b}_\preceq = \omega_1, \quad \mathfrak{d}_\preceq = \mathfrak{c}$.

Proof. Inequalities $\mathfrak{b}_\preceq \geq \omega_1$ and $\mathfrak{d}_\preceq \leq \mathfrak{c}$ are obvious. To show the other inequalities we first construct a family $\mathcal{P} \subseteq (\omega)^\omega_\omega$ of cardinality $\mathfrak{c}$ such that for every two partitions $P_1, P_2 \in \mathcal{P}$ if $p_1 \in P_1$ and $p_2 \in P_2$ then $p_1 \cap p_2$ is finite.

We deal with partitions of $\mathbb{Z} \times \mathbb{Z}$ instead of partitions of ω. Let $p_i^\alpha = \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z} : i \leq z_2 - \alpha z_1 < i+1\}$ for $i \in \mathbb{Z}$ and $\alpha \in [0, +\infty)$. Then $P^\alpha = \{p_i^\alpha : i \in \mathbb{Z}\}$ is a partition from $(\mathbb{Z}\times\mathbb{Z})^\omega_\omega$. It is not difficult to check that a family $\mathcal{P} = \{P^\alpha : \alpha \in [0, +\infty)\}$ has a needed property.

Now, if $\mathcal{R} \subseteq (\omega)^\omega_\omega$ is any subfamily of $\mathcal{P}$ of size ω_1 then $\mathcal{R}$ cannot be dominated by one partition. Indeed, if there exists a partition $P \in (\omega)^\omega_\omega$ such that for every $R \in \mathcal{R}$ and every $r \in R$ we have an element $p \in P$ such that $p \subseteq r$ then we get a contradiction as for different $R_1, R_2 \in \mathcal{R}$ and $r_1 \in R_1, r_2 \in R_2$ there is no $p \in P$ which is simultaneously contained in r_1 and r_2.

On the other hand, let us consider a family $\mathcal{R}$ such that every partition from $(\omega)^\omega_\omega$ is dominated by a partition from $\mathcal{R}$. For a given $R \in \mathcal{R}$ we define $\mathcal{P}_R = \{P \in \mathcal{P} : (\forall p \in P)(\exists r \in R) r \subseteq p\}$. Obviously $\mathcal{P} = \bigcup_{R\in\mathcal{R}} \mathcal{P}_R$. Moreover, every family $\mathcal{P}_R$ is at most countable because any element of R cannot be contained in elements of different partitions from $\mathcal{P}_R$. Therefore

$$\mathfrak{c} \leq |\mathcal{P}| \leq \omega \cdot |\mathcal{R}|$$

and we are done. □

Theorem 2.5 $\operatorname{add}_t(\mathbb{S}_2) = \omega_1, \quad \operatorname{cof}_t(\mathbb{S}_2) = \mathfrak{c}$.

Proof. As $\omega_1 \leq \operatorname{add}_t(\mathbb{S}_2)$ and $\operatorname{cof}_t(\mathbb{S}_2) \leq \mathfrak{c}$ then thanks to Lemma 2.4 we have to prove only $\operatorname{add}_t(\mathbb{S}_2) \leq \mathfrak{b}_\preceq$ and $\operatorname{cof}_t(\mathbb{S}_2) \geq \mathfrak{d}_\preceq$.

We observe the following useful fact. Let $\mathcal{P} \subseteq (\omega)^\omega_\omega$ be a family of partitions and $\mathcal{A} \subseteq \mathbb{S}_2$. Let us assume that for every partition $P \in \mathcal{P}$ there exist $A_P \in \mathcal{A}$ and $x_P \in 2^\omega$ such that $\bigcup_{p\in P}[\mathbf{0}_p] \subseteq A_P + x_P$, where $\mathbf{0}_p$ denotes a function constantly equal to 0 on its domain, which is the set p. Then there exists a family $\mathcal{R} \subseteq (\omega)^\omega_\omega$ of size $|\mathcal{A}|$ such that for every $P \in \mathcal{P}$ there exists $R \in \mathcal{R}$ such that $P \preceq R$. Indeed, without loss of generality we can assume that $A_P = \bigcup_{i<\omega}[f_i^P]$, where $\{f_i^P : i < \omega\} \subseteq Pif$ and

$\{\mathrm{dom}(f_i^P) : i < \omega\} \in (\omega)_\omega^\omega$ and by Lemma 1.2 we get that for every $p \in P$ there exists a natural number i_p such that $[\mathbf{0}_p] \subseteq [f_{i_p}^P + x_P \restriction \mathrm{dom}(f_{i_p}^P)]$. Thus $\mathrm{dom}(f_{i_p}^P) \subseteq p$ and, consequently, $P \preceq \{\mathrm{dom}(f_i^P) : i < \omega\}$. Hence $\mathcal{R} = \{\{\mathrm{dom}(f_i^P) : i < \omega\} : P \in \mathcal{P}\}$ is a family of the sort we are looking for.

Now, let $\mathcal{P} \subseteq (\omega)_\omega^\omega$ be an arbitrary family of partitions of size less than $\mathrm{add}_t(\mathbb{S}_2)$. From the definition of $\mathbb{S}_2$ we obtain that our assumption is fulfilled for a family $\mathcal{A}$ having one element. Thus $\mathcal{P}$ is bounded by one partition and we get $\mathrm{add}_t(\mathbb{S}_2) \leq \mathfrak{b}_\preceq$.

On the other hand, our assumption is fulfilled also for $\mathcal{P} = (\omega)_\omega^\omega$ and $\mathcal{A} \subseteq \mathbb{S}_2$ being a transitive base for $\mathbb{S}_2$. In this situation, the family $\mathcal{R}$ obtained from the fact mentioned above is a dominating family of partitions, so we have $\mathrm{cof}_t(\mathbb{S}_2) \geq \mathfrak{d}_\preceq$, which ends the proof. □

The last transitive property we deal with is translatability. In 1993 Carlson in [4] introduced the notion of κ–translatability and proved that the σ-ideal of meagre subsets of the real line and the σ-ideal generated by closed null subsets of the real line are ω–translatable. Bartoszyński in [1] proved that the σ-ideal of null subsets of the Cantor space is not 2–translatable. Kysiak in [11] introduced a natural notion of a translatability number.

As far as $\mathbb{S}_2$ is concerned, its translatability number can be computed precisely.

Theorem 2.6 $\tau(\mathbb{S}_2) = \omega_1$.

Proof. To begin with, we show that $\mathbb{S}_2$ is ω–translatable. Let $A \in \mathbb{S}_2$ be arbitrary. As usual, without loss of generality we can assume that $A = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)_\omega^\omega$. For every $i < \omega$ let us fix a partition $P_i = \{p_{ij} : j < \omega\} \in (\mathrm{dom}(f_i))_\omega^\omega$. Then $\{p_{ij} : i, j < \omega\} \in (\omega)_\omega^\omega$. We define

$$B = \bigcup_{i<\omega} \bigcup_{j<\omega} [\mathbf{0}_{p_{ij}}].$$

Obviously, $B \in \mathbb{S}_2$. For every $T = \{t_j : j < \omega\} \in [2^\omega]^\omega$ we define $g \in 2^\omega$ as follows:

$$(\forall i, j < \omega)\, g \restriction p_{ij} = (f_i + t_j) \restriction p_{ij}.$$

It is a routine calculation to show that $A + T \subseteq B + g$.

To show the other inequality, let us consider first a partition P of ω into infinite parts. We can observe that there exists a set $T \in [2^\omega]^{\omega_1}$ such that for every family $\{h_i : i < \omega\} \subseteq Pif$ if $\{\mathrm{dom}(h_i) : i < \omega\} = P$ then $T \not\subseteq \bigcup_{i<\omega}[h_i]$. Namely, it is enough to take T such that $(\forall p \in P)(\forall x, y \in T)(x \neq y \Rightarrow x \restriction p \neq y \restriction p)$.

Let $A = \{\mathbf{0}_\omega\}$. We claim that this set witnesses that $\mathbb{S}_2$ is not ω_1–translatable. So suppose $B = \bigcup_{i<\omega}[h_i]$ where $\{h_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(h_i) : i < \omega\} = P \in (\omega)_\omega^\omega$. Consider the set T defined as above. Then no translation of B covers $T = A + T$. □

3. Transitive operations on $\mathbb{S}_2$. In this paragraph we apply transitive operations to the ideal $\mathbb{S}_2$. To begin with, let us recall some definitions.

Let us assume that $\mathcal{J}$ is a σ-ideal of subsets of an abelian group G which is proper, translation invariant, symmetric and contains all singletons. We define (cf. [19])

$$s(\mathcal{J}) = \{A \subseteq G : (\forall B \in \mathcal{J})\, A + B \neq G\},$$
$$g(\mathcal{J}) = \{A \subseteq G : (\forall B \in \mathcal{J})\, A + B \in \mathcal{J}\}$$

(Seredyński used $\mathcal{J}^*$ instead of $s(\mathcal{J})$). In [19] many basic properties of operations s and g can be found. If we apply these operations to the σ-ideals of meagre sets $\mathcal{M}$ and of null sets $\mathcal{N}$ we obtain strongly null sets $s(\mathcal{M})$, strongly meagre sets $s(\mathcal{N})$, meagre-additive sets $g(\mathcal{M})$ and null-additive sets $g(\mathcal{N})$ (see [2] for more information).

The following are well-known.

Fact 3.1 $\quad \mathrm{non}(s(\mathcal{J})) = \mathrm{cov}_t(\mathcal{J}), \qquad \mathrm{non}(g(\mathcal{J})) = \mathrm{add}_t^*(\mathcal{J}).$ □

We can also observe other basic relations.

Proposition 3.2 $\quad \mathrm{cov}_t(s(\mathcal{J})) \geq \mathrm{non}(\mathcal{J}), \qquad \mathrm{add}_t^*(g(\mathcal{J})) = \mathrm{non}(g(\mathcal{J})).$

Proof. Straightforward from definitions. □

We prove now that σ-ideals $\mathbb{S}_2$ and $\mathbb{B}_2$ are closely related to each other.

Theorem 3.3 $\quad s(\mathbb{S}_2) = \mathbb{B}_2$.

Proof. Let us consider any $A \subseteq 2^\omega$. A standard calculation shows that if for some $X \in [\omega]^\omega$ we have $A \restriction X = 2^X$ then $A + [\mathbf{0}_X] = 2^\omega$. Hence if $A \notin \mathbb{B}_2$ then $A \notin s(\mathbb{S}_2)$.

On the other hand, let us consider any $C \subseteq 2^\omega$ such that $B + C = 2^\omega$ for some $B \in \mathbb{S}_2$. As in proofs in Paragraph 2, without loss of generality we can assume that $B = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. Then there exists $i < \omega$ such that $C \restriction \mathrm{dom}(f_i) = 2^{\mathrm{dom}(f_i)}$. Indeed, if we suppose that for all $i < \omega$ there exists $g_i \in 2^{\mathrm{dom}(f_i)} \setminus (C \restriction \mathrm{dom}(f_i))$ then we have $\bigcup_{i<\omega}(f_i + g_i) \in 2^\omega \setminus (B + C)$. Thus if $C \notin s(\mathbb{S}_2)$ then $C \notin \mathbb{B}_2$ which completes the proof. □

In [7] the authors showed that the covering number of $\mathbb{B}_2$ is a weird object and it is difficult to find reasonable estimations for it. In particular, it is relatively consistent that Martin's Axiom holds, $\mathfrak{c} = \omega_2$ and $\mathrm{cov}(\mathbb{B}_2) = \omega_1$. The following corollary shows that the situation for the transitive covering number of $\mathbb{B}_2$ is different.

Corollary 3.4 *If Martin's Axiom holds then* $\mathrm{cov}_t(\mathbb{B}_2) = \mathfrak{c}$.

Proof. From Theorem 3.3 and Proposition 3.2 we obtain that $\mathrm{cov}_t(\mathbb{B}_2) \geq \mathrm{non}(\mathbb{S}_2)$. It was proved in [5] that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ and it is well-known that under Martin's Axiom we have $\aleph_0\text{-}\mathfrak{s} = \mathfrak{c}$. □

In order to describe $g(\mathbb{S}_2)$ we need to introduce more definitions. By Inj we denote the set of all injections from ω into ω. For $A \subseteq 2^\omega$ and $\varphi \in Inj$ we put $\varphi * A = \{x \circ \varphi : x \in A\}$ and $A_\varphi = \{x \in 2^\omega : x \circ \varphi \in A\}$. It is easy to observe that we have $\varphi * A_\varphi = A$ and

$A \subseteq (\varphi * A)_\varphi$. Let $\mathcal{J}$ be a σ-ideal of subsets of 2^ω. We say that $\mathcal{J}$ is productive if for every $A \subseteq 2^\omega$ and $\varphi \in Inj$ if $\varphi * A$ is in $\mathcal{J}$ then so is A. We say that $\mathcal{J}$ has WFP (Weak Fubini Property) if for every $A \subseteq 2^\omega$ and $\varphi \in Inj$ if A_φ is in $\mathcal{J}$ then so is A. Straight from the definitions we obtain that $\mathbb{S}_2$, σ-ideals of meagre and null sets are productive and have WFP. For more discussion on these properties cf. [9].

We put

$$p(\mathcal{J}) = \{A \subseteq 2^\omega : (\forall \varphi \in Inj)\, \varphi * A \in \mathcal{J}\}.$$

In other words, $A \in p(\mathcal{J})$ if for every $T \in [\omega]^\omega$ the set $A \restriction T$ is in $\mathcal{J}(T)$, where $\mathcal{J}(T)$ denotes a version of $\mathcal{J}$ defined on 2^T instead of 2^ω.

Theorem 3.5 $g(\mathbb{S}_2) = p(\mathbb{S}_2)$.

Proof. Let us assume that $A \in g(\mathbb{S}_2)$ that is $(\forall B \in \mathbb{S}_2) A + B \in \mathbb{S}_2$. It is not difficult to observe that this condition is equivalent to $(\forall T \in [\omega]^\omega)[\mathbf{0}_T] + A \in \mathbb{S}_2$. But we can prove that if $\varphi \in Inj$ then $[\mathbf{0}_{\mathrm{rng}(\varphi)}] + A = (\varphi * A)_\varphi$. Hence, reformulating our condition we obtain $(\forall \varphi \in Inj)\, (\varphi * A)_\varphi \in \mathbb{S}_2$. Thus, as $\mathbb{S}_2$ is productive and has WFP, we show that this fact is equivalent to $(\forall \varphi \in Inj)\, \varphi * A \in \mathbb{S}_2$ and, consequently, to $A \in p(\mathbb{S}_2)$. $\square$

Finally, we will show that all operations that appeared in this paragraph are versions of one operation, defined in [19].

Let $\mathcal{A}, \mathcal{B}$ be translation invariant families of subsets of a group G. We put

$$\mathcal{G}_t(\mathcal{A}, \mathcal{B}) = \{A \subseteq G : (\forall B \in \mathcal{B})\, A + B \in \mathcal{A}\}.$$

Then we have the following results.

Proposition 3.6 *Let $\mathcal{J}$ be a translation invariant, symmetric σ-ideal of subsets of a group G. Then*
(a) $s(\mathcal{J}) = \mathcal{G}_t(\mathcal{P}(G) \setminus \{G\}, \mathcal{J})$,
(b) $g(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathcal{J})$.
If $G = 2^\omega$ and $\mathcal{J}$ is productive and has WFP then
(c) $p(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathbb{S}_2)$.

Proof. (a) and (b) are reformulations of definitions and were observed in [19]. To prove (c) it is enough to repeat carefully the proof of Theorem 3.5. $\square$

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