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ON SOME NEW IDEALS ON THE CANTOR AND BAIRE SPACES

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ABSTRACT. We define and investigate some new ideals of subsets of the Cantor space and the Baire space. We show that combinatorial properties of these ideals can be described by the splitting and reaping cardinal numbers. We show that there exist perfect Luzin sets for these ideals on the Baire space.

0. INTRODUCTION

For each infinite subset T of the set ω of all natural numbers let us denote by $\mathcal{K}(T)$ the σ-ideal of meagre subsets of the space 2^T with the canonical product topology. By $\mathcal{L}(T)$ we denote the σ-ideal of Lebesgue measure zero subsets of 2^T with respect to the canonical product measure.

Notice that if T is a subset of ω then we can identify the spaces $2^T \times 2^{\omega\setminus T}$ and the Cantor space 2^ω using the canonical homeomorphism π_T defined by $\pi_T(x) = (x|T, x|(\omega \setminus T))$. Directly from the definition of meagre sets it follows that if $A \in \mathcal{K}(T)$ then $A \times 2^{\omega\setminus T} \in \mathcal{K}(\omega)$. The same observation is also true for the ideal $\mathcal{L}(\omega)$ but it is evidently false for the σ-ideal of all countable subsets of the Cantor space. We call this property of the ideals $\mathcal{K}(\omega)$ and $\mathcal{L}(\omega)$ *productivity.*

There are other natural productive σ-ideals of subsets of the Cantor space, e.g. the σ-ideal $\mathcal{K}(\omega) \cap \mathcal{L}(\omega)$. It is interesting that among them there exists the least productive σ-ideal which contains all points. We call this ideal S_2. There exists also the least productive ideal of subsets of 2^ω and we call it I_2. These ideals have Borel bases but they do not satisfy the countable chain condition - there exists a family of continuum many pairwise disjoint Borel sets outside the ideal S_2. The ideal I_2 is not σ-additive and the ideal S_2 is precisely σ-additive. The minimum cardinality of bases of these ideals is continuum. The covering number *cov* of both ideals is equal to the *reaping* cardinal $\mathfrak{r}$ and the last basic combinatorial invariants (cardinal numbers *non*) are described in terms of *splitting* cardinal numbers. These results show that both splitting and reaping cardinals are closely connected with natural mathematical objects on the classical Cantor space.

We also consider the minimal productive σ-ideal S_ω of subsets of the Baire space ω^ω. We show that there exists an uncountable closed subset P of ω^ω which is a

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Luzin set for S_ω, i.e. the intersection of the set P with any set from S_ω is countable. This fact completely determines the basic combinatorial invariants of this ideal.

1. Notation, definitions and basic observations

In this paper we use the standard set theoretical notation. For example, ω denotes the first infinite cardinal number which we shall identify with the set of all natural numbers. The cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. If κ is a cardinal number then $[X]^\kappa$ denotes the family of all subsets of X of cardinality κ and $[X]^{<\kappa}$ denotes the family of all subsets of X of cardinality strictly less then κ. $X^{<\omega}$ denotes the set of all finite sequences of elements of the set X. The power set of a set X is denoted by $P(X)$. For $A, B \subseteq \omega$ we put $A \subseteq^* B$ if and only if $card\,(A \setminus B) < \omega$.

If X is a discrete topological space then we endow X^ω with the standard product topology. In particular, for $X = 2$ and $X = \omega$ we get the Cantor space and the Baire space, respectively.

For an ideal $\mathcal{J}$ of subsets of X we consider the following cardinal numbers

$$\begin{aligned}
\operatorname{add}(\mathcal{J}) &= \min\{card\,(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{J} \;\;\&\;\; \bigcup \mathcal{A} \notin \mathcal{J}\},\\
\operatorname{cov}(\mathcal{J}) &= \min\{card\,(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{J} \;\;\&\;\; \bigcup \mathcal{A} = X\},\\
\operatorname{non}(\mathcal{J}) &= \min\{card\,(B) : B \subseteq X \;\;\&\;\; B \notin \mathcal{J}\},\\
\operatorname{cof}(\mathcal{J}) &= \min\{card\,(\mathcal{A}) : \mathcal{A} \subseteq \mathcal{J} \;\;\&\;\; (\forall A \in \mathcal{J})(\exists B \in \mathcal{A})(A \subseteq B)\}.
\end{aligned}$$

Note that if $\mathcal{J}$ is a proper ideal and $\bigcup \mathcal{J} = X$ then the following relations hold:

$$\operatorname{add}(\mathcal{J}) \le \operatorname{cov}(\mathcal{J}),\; \operatorname{add}(\mathcal{J}) \le \operatorname{non}(\mathcal{J}),\; \operatorname{cov}(\mathcal{J}) \le \operatorname{cof}(\mathcal{J}),\; \operatorname{non}(\mathcal{J}) \le \operatorname{cof}(\mathcal{J}).$$

Suppose that $\mathcal{J}$ is an ideal of subsets of X and $\mathcal{A} \subseteq P(X)$. We say that $\mathcal{J}$ has an $\mathcal{A}$-*base* if for each $A \in \mathcal{J}$ there exists such $B \in \mathcal{A} \cap \mathcal{J}$ that $A \subseteq B$. Hence, in particular, if X is a topological space then $\mathcal{J}$ has an F_σ-base if each element from $\mathcal{J}$ can be covered by some F_σ subset of X from $\mathcal{J}$.

Let $\mathcal{K}$ and $\mathcal{L}$ denote the σ-ideals of meagre subsets and of Lebesgue measure zero subsets of the Cantor space 2^ω, respectively. The ideal $\mathcal{K}$ has an F_σ-base and the ideal $\mathcal{L}$ has a G_δ-base.

From now on let us assume that X has at least two elements. We define

$$Pif(X) = \{\varphi : \;\varphi \text{ is a function} \;\;\&\;\; \operatorname{dom}(\varphi) \in [\omega]^\omega \;\;\&\;\; \operatorname{rng}(\varphi) \subseteq X\}.$$

If $\varphi \in Pif(X)$ then we put

$$\begin{aligned}
[\varphi]_X &= \{x \in X^\omega : \;\varphi \subseteq x\},\\
[\varphi]^*_X &= \{x \in X^\omega : \;\varphi \subseteq^* x\}.
\end{aligned}$$

If we treat X as a discrete topological space then $[\varphi]_X$ is a closed and $[\varphi]^*_X$ is an F_σ subset of X^ω for each $\varphi \in Pif(X)$.

Now we are able to define the ideals we are going to deal with. Let I_X and I^*_X denote the ideals generated by families $\{[\varphi]_X : \;\varphi \in Pif(X)\}$ and $\{[\varphi]^*_X : \;\varphi \in$

$Pif(X)\}$, respectively. Then $[X]^{<\omega} \subseteq I_X \subseteq I_X^*$ and I_X^* is a proper ideal of subsets of X^ω. The first ideal has a closed base and the other one has an F_σ-base.

Similarly, let S_X and S_X^* denote the σ-ideals generated by families $\{[\varphi]_X : \varphi \in Pif(X)\}$, $\{[\varphi]_X^* : \varphi \in Pif(X)\}$, respectively. It is easy to observe that $S_X = S_X^*$. Hence we have $I_X \subseteq I_X^* \subseteq S_X$. The ideal S_X has an F_σ-base and is proper. Directly from the definition of the ideals we can deduce that

$$\mathrm{cov}(I_X) = \mathrm{cov}(I_X^*) = \mathrm{cov}(S_X).$$

If $j : \omega \to \omega$ is an injection and $A \subseteq 2^\omega$ then we define

$$j * A = \{x \in 2^\omega : x \circ j \in A\}.$$

We say that an ideal $\mathcal{J}$ of subsets of 2^ω is *productive* if $j * A \in \mathcal{J}$ for each $A \in \mathcal{J}$ and any injection $j : \omega \to \omega$. As we have mentioned in the introduction, the ideals $\mathcal{K}$ and $\mathcal{L}$ are productive.

Let $\varphi \in Pif(2)$ and let $j : \omega \to \mathrm{dom}(\varphi)$ be any bijection. Then $j * \{\varphi \circ j\} = [\varphi]_2$. Conversely, if $j : \omega \to \omega$ is an injection and $x \in 2^\omega$, then $x \circ j^{-1} \in Pif(2)$ and

$$j * \{x\} = [x \circ j^{-1}]_2.$$

These observations imply that I_2 and S_2 are the least productive ideal and σ-ideal of subsets of 2^ω (containing all points), respectively. Since $\mathcal{K}$ and $\mathcal{L}$ are productive, we see that $S_2 \subseteq \mathcal{K} \cap \mathcal{L}$.

Let us recall that σ-ideals of subsets of a Polish space with Borel bases have plenty of interesting properties (see e.g. [2]). The next result shows the main difference between the ideal S_2 and the ideals $\mathcal{K}$ and $\mathcal{L}$.

Theorem 1.1. *There exists a family $\mathcal{F}$ of pairwise disjoint Borel subsets of 2^ω such that $card(\mathcal{F}) = \mathfrak{c}$ and none of elements of $\mathcal{F}$ belongs to S_2.*

Proof. For each real number $\alpha \in [0, 1]$ we put

$$A_\alpha = \{x \in 2^\omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} x(i) = \alpha\}.$$

It is easy to check that $\{A_\alpha : \alpha \in [0, 1]\}$ is the required family. □

Let A, S be two infinite subsets of ω. We say that S *splits* A if $card(A \cap S) = card(A \setminus S) = \omega$. Let us recall the following three cardinal numbers (see e.g. [3]):

$$\mathfrak{s} = \min\{card(\mathcal{S}) : \mathcal{S} \subseteq [\omega]^\omega \ \& \ (\forall A \in [\omega]^\omega)(\exists S \in \mathcal{S})(S \text{ splits } A)\},$$

$$\aleph_0\text{-}\mathfrak{s} = \min\{card(\mathcal{S}) : \mathcal{S} \subseteq [\omega]^\omega \ \& \ (\forall \mathcal{A} \in [[\omega]^\omega]^\omega)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S \text{ splits } A)\},$$

$$\mathfrak{r} = \min\{card(\mathcal{R}) : \mathcal{R} \subseteq [\omega]^\omega \ \& \ (\forall S \in [\omega]^\omega)(\exists R \in \mathcal{R})(S \text{ does not split } R)\}.$$

The cardinal numbers $\mathfrak{s}$ and $\mathfrak{r}$ are called *splitting* and *reaping*, respectively. It is easy to check that $\omega_1 \le \mathfrak{s} \le \aleph_0\text{-}\mathfrak{s} \le \mathfrak{c}$ and $\omega_1 \le \mathfrak{r} \le \mathfrak{c}$. It is an open problem now if $\mathfrak{s} = \aleph_0\text{-}\mathfrak{s}$ can be proved in ZFC. An easy reformulation of the definition of the reaping number $\mathfrak{r}$ gives us the following description

$$\mathfrak{r} = \min\{card(\mathcal{R}) : \mathcal{R} \subseteq [\omega]^\omega \ \& \ (\forall S \in [\omega]^\omega)(\exists R \in \mathcal{R})(R \subseteq S \ \vee \ R \subseteq \omega \setminus S)\}.$$

Let us introduce an auxiliary cardinal number. Namely, we define

$$\text{fin-}\mathfrak{s} = \min\{card(\mathcal{S}) : \mathcal{S} \subseteq [\omega]^\omega \ \& \ (\forall \mathcal{A} \in [[\omega]^\omega]^{<\omega})(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S \text{ splits } A)\}.$$

Lemma 1.2. $\mathfrak{s} = \text{fin-}\mathfrak{s}$.

Proof. It is clear that $\mathfrak{s} \leq \text{fin-}\mathfrak{s}$. Suppose now that $\mathcal{S} \subseteq [\omega]^\omega$ is such that $card\,(\mathcal{S}) = \mathfrak{s}$ and every infinite subset $A \subseteq \omega$ is split by some element of $\mathcal{S}$. We may assume that $\mathcal{S}$ is a field of subsets of ω containing all finite sets. One can show by an easy induction on $n \in \omega$ that

$$(\forall \mathcal{A} \in [[\omega]^\omega]^n)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S \textit{ splits } A)\}.$$

□

2. Basic properties

Assume that $X \subseteq Y$, $card\,(X) \geq 2$. Then it is easy to notice that

$$I_X = I_Y \cap P(X^\omega), \quad I^*_X = I^*_Y \cap P(X^\omega) \quad \text{and} \quad S_X = S_Y \cap P(X^\omega).$$

Let us recall a well-known fact.

Lemma 2.1. *Suppose that $\mathcal{I}$ is an ideal of subsets of X, $\mathcal{J}$ is an ideal of subsets of Y, $X \subseteq Y$ and $\mathcal{I} = \mathcal{J} \cap P(X)$. Then* $\mathrm{cov}(\mathcal{I}) \leq \mathrm{cov}(\mathcal{J})$ *and* $\mathrm{non}(\mathcal{J}) \leq \mathrm{non}(\mathcal{I})$.

For each set X we have

Lemma 2.2. $\mathrm{add}(I_X) = \mathrm{non}(I_X) = \omega$.

Proof. Lemma 2.1 implies that we have to prove $\mathrm{non}(I_2) = \omega$ only. Let

$$A = \{x \in 2^\omega : \, (\forall^\infty n \in \omega)\, x(n) = 0\}.$$

Then $card\,(A) = \omega$ and it is easy to check that $A \notin I_2$. □

We call a family $\mathcal{F} \subseteq Pif(X)$ *normal* if for each two different $\varphi_1, \varphi_2 \in \mathcal{F}$ we have $\mathrm{dom}(\varphi_1) \cap \mathrm{dom}(\varphi_2) = \emptyset$. Notice that if $\{\varphi_i : i \in I\} \subseteq Pif(X)$ and $card\,(X) \leq \omega$ then there exists such a normal family $\{\psi_i : i \in I\}$ that $\psi_i \subseteq \varphi_i$ for each $i \in I$. Notice also that if $\varphi, \psi \in Pif(X)$ and $\psi \subseteq \varphi$ then $[\varphi]_X \subseteq [\psi]_X$ and $[\varphi]^*_X \subseteq [\psi]^*_X$. Therefore for every $A \in X^\omega$ we have

1) $A \in I_X \iff A \subseteq \bigcup_{\varphi \in \mathcal{F}} [\varphi]_X$ for some finite normal family $\mathcal{F} \subseteq Pif(X)$,
2) $A \in I^*_X \iff A \subseteq \bigcup_{\varphi \in \mathcal{F}} [\varphi]^*_X$ for some finite normal family $\mathcal{F} \subseteq Pif(X)$,
3) $A \in S_X \iff A \subseteq \bigcup_{\varphi \in \mathcal{F}} [\varphi]_X$ for some countable normal family $\mathcal{F} \subseteq Pif(X)$.

Lemma 2.3. *Suppose that $\{\varphi_i : i \in I\}$ is a normal family of functions from $Pif(X)$, $\varphi \in Pif(X)$ and*

$$[\varphi]_X \subseteq \bigcup_{i \in I} [\varphi_i]_X \quad ([\varphi]^*_X \subseteq \bigcup_{i \in I} [\varphi_i]^*_X).$$

Then $[\varphi]_X \subseteq [\varphi_i]_X$ $([\varphi]^*_X \subseteq [\varphi_i]^*_X)$ *for some* $i \in I$.

Proof. We shall prove only the second case of the lemma. Suppose that $[\varphi]^*_X \not\subseteq [\varphi_i]^*_X$ for each $i \in I$, i.e $\varphi_i \not\subseteq^* \varphi$ for each $i \in I$. Hence, we may find a family $\{K_i : i \in I\}$ of infinite subsets of ω such that

1) $K_i \subseteq \mathrm{dom}(\varphi_i)$,
2) $(\forall k \in K_i)(k \notin \mathrm{dom}(\varphi) \ \vee \ \varphi_i(k) \neq \varphi(k))$

for each $i \in I$. Notice that elements from the family $\{K_i : i \in I\}$ are pairwise disjoint. We consider an arbitrary function $f \in X^\omega$ such that $\varphi \subseteq f$ and $f(k) \neq \varphi_i(k)$ for each $k \in K_i \setminus \mathrm{dom}(\varphi)$ and $i \in I$. One can show with ease that $f \in [\varphi]^*_X \setminus \bigcup_{i\in I}[\varphi_i]^*_X$, which leads to contradiction.
The proof of the first case of the lemma is similar to the presented one: infinite sets K_i should be replaced by singletons. □

Let $\mathcal{I}$ be an arbitrary ideal and let $\kappa \leq \lambda$ be two infinite cardinals. A family $\mathcal{A} \subseteq \mathcal{I}$ is called a (κ, λ)-*family* for $\mathcal{I}$ if $card(\mathcal{A}) = \lambda$ and $card(\{A \in \mathcal{A} : A \subseteq S\}) < \kappa$ for each $S \in \mathcal{I}$.

Lemma 2.4. *Let* $card(X) \geq 2$. *Then there exists a family* $\mathcal{A} \subseteq I_X$ *which is an* $(\omega, \mathfrak{c})$-*family for* I^*_X *and an* $(\omega_1, \mathfrak{c})$-*family for* S_X.

Proof. We may assume that $\{0, 1\} \subseteq X$. Let us fix a family $\mathcal{F} \subseteq [\omega]^\omega$ of cardinality $\mathfrak{c}$ such that $card(A \cap B) < \omega$ for any two different $A, B \in \mathcal{F}$. Let $\mathcal{A} = \{[A \times \{1\}]_X : A \in \mathcal{F}\}$. It is clear that $\mathcal{A} \subseteq I_X$. Suppose now that $S \in I^*_X$. Then for some $k \in \omega$ and a normal family $\{\varphi_1, \ldots, \varphi_k\} \subseteq Pif(X)$ we have $S \subseteq [\varphi_1]^*_X \cup \cdots \cup [\varphi_k]^*_X$. Let $\mathcal{Y} = \{A \in \mathcal{F} : [A \times \{1\}]^*_X \subseteq [\varphi_1]^*_X \cup \cdots \cup [\varphi_k]^*_X\}$. We claim that $card(\mathcal{Y}) \leq k$. Suppose otherwise. Notice that Lemma 2.3 implies

$$\mathcal{Y} = \{A \in \mathcal{F} : (\exists i \in \{1, \ldots, k\})([A \times \{1\}]^*_X \subseteq [\varphi_i]^*_X)\}.$$

Therefore, there are two different $A, B \in \mathcal{F}$ and $i \in \{1, \ldots, k\}$ such that $[A \times \{1\}]^*_X \subseteq [\varphi_i]^*_X, [B \times \{1\}]^*_X \subseteq [\varphi_i]^*_X$. But then $\varphi_i \subseteq^* A \times \{1\}$ and $\varphi_i \subseteq^* B \times \{1\}$ hence $card(A \cap B) = \omega$. So we obtained a contradiction. The proof of the other part of the lemma is similar. □

It is easy to check (see e.g. [1]) that if there exists a (κ, λ)-family for an ideal $\mathcal{I}$ and $\kappa < \lambda$ (or $\kappa = \lambda$ and κ is regular) then $\mathrm{add}(\mathcal{I}) \leq \kappa$ and $\mathrm{cof}(\mathcal{I}) \geq \lambda$. Hence we obtained the following result.

Theorem 2.5. *Let* $card(X) \geq 2$. *Then*

$$\mathrm{add}(I^*_X) = \omega, \ \mathrm{add}(S_X) = \omega_1.$$

If moreover $card(X) \leq \omega$ *then*

$$\mathrm{cof}(I_X) = \mathrm{cof}(I^*_X) = \mathrm{cof}(S_X) = \mathfrak{c}.$$

3. Ideals on the Baire space

In this part we shall discuss the properties of ideals I_ω, I^*_ω and S_ω. Hence we shall work now on the classical Baire space of infinite sequences of natural numbers. Let us recall that if $\mathcal{I}$ is an ideal of subsets of X then a set $L \subseteq X$ is called a *Luzin* set for $\mathcal{I}$ if $card\,(L) = card\,(X)$ and $card\,(A \cap L) \leq \omega$ for each $A \in \mathcal{I}$. Notice if there exists a Luzin set for an ideal $\mathcal{I}$ then $\mathrm{non}(\mathcal{I}) \leq \omega_1$ and $\mathrm{cov}(\mathcal{I}) = card\,(X)$.

A subset of a topological space is *perfect* if it is closed and contains no isolated points.

Theorem 3.1. *There exists a perfect Luzin set for the ideal S_ω.*

Proof. Fix a bijection $b : 2^{<\omega} \to \omega$. To each $f \in 2^\omega$ associate $\tilde{f} : \omega \to \omega$ defined by $\tilde{f}(n) = b(f(0), \ldots, f(n-1))$. Then $\{\tilde{f} : f \in 2^\omega\}$ is a perfect set and it is Luzin for S_ω because no two of its members agree infinitely often. □

Putting together Lemma 2.2, Theorem 2.5, Theorem 3.1 and the observations from the beginning of this part we are able to describe cardinal coefficients add, non, cov, cof of the ideals $I_\omega, I^*_\omega, S_\omega$.

Theorem 3.2.

1) $\mathrm{add}(I_\omega) = \mathrm{add}(I^*_\omega) = \omega < \mathrm{add}(S_\omega) = \omega_1$;
2) $\mathrm{non}(I_\omega) = \mathrm{non}(I^*_\omega) = \omega < \mathrm{non}(S_\omega) = \omega_1$;
3) $\mathrm{cov}(I_\omega) = \mathrm{cov}(I^*_\omega) = \mathrm{cov}(S_\omega) = \mathfrak{c}$;
4) $\mathrm{cof}(I_\omega) = \mathrm{cof}(I^*_\omega) = \mathrm{cof}(S_\omega) = \mathfrak{c}$.

4. Ideals on the Cantor spaces

In this section we shall discuss the ideals I_n, I^*_n and S_n for natural numbers $n \geq 2$. Let us recall we indentify a number n with the set $\{0, \ldots, n-1\}$. Hence, for example, the ideals I_2, I^*_2 and S_2 are ideals of subsets of the classical Cantor space.

Lemma 4.1. *Let $2 \leq n \leq \omega$. Then $\mathrm{cov}(I_n) = \mathrm{cov}(I^*_n) = \mathrm{cov}(S_n) = \mathfrak{r}$.*

Proof. As we have observed in the introduction the cardinal numbers $\mathrm{cov}(I_n)$, $\mathrm{cov}(I^*_n)$ and $\mathrm{cov}(S_n)$ are equal. Hence we shall prove $\mathrm{cov}(S_n) = \mathfrak{r}$.

Suppose first $\mathcal{F} \subseteq Pif(2)$ is such a family that

$$\bigcup_{\varphi \in \mathcal{F}} [\varphi] = 2^\omega.$$

Let $\mathcal{R} = \{\varphi^{-1}[\{i\}] : i \in \{0,1\} \;\&\; \varphi \in \mathcal{F}\} \cap [\omega]^\omega$. Then

$$(\forall A \in [\omega]^\omega)(\exists R \in \mathcal{R})(R \subseteq A \;\vee\; R \subseteq A^c),$$

so $\mathrm{cov}(S_2) \geq \mathfrak{r}$. We get from Lemma 2.1 that $\mathrm{cov}(S_n) \geq \mathfrak{r}$ for each $n \geq 2$.
Let us fix now such a family $\mathcal{R} \subseteq [\omega]^\omega$ of cardinality $\mathfrak{r}$ that for each $A \in [\omega]^\omega$ there exists $R \in \mathcal{R}$ which is contained either in A or in its complement. We can inscribe an isomorphic copy of the whole family $\mathcal{R}$ into each of its elements and repeat this

process countably many times. Then we obtain a family $\mathcal{R}^* \subseteq [\omega]^\omega$ of cardinality $\mathfrak{r}$ such that for each partition of ω into finitely many pieces there exists $R \in \mathcal{R}^*$ which is contained in some element of the partition. Let

$$\mathcal{F} = \{[R \times \{i\}] : i \in \{1, \dots, n\} \ \& \ R \in \mathcal{R}^*\}.$$

Then $\mathcal{F} \subseteq S_n$ and $\bigcup \mathcal{F} = n^\omega$. Hence $\mathrm{cov}(S_n) \leq \mathfrak{r}$. □

Lemma 4.2. *Let* $2 \leq n \leq \omega$. *Then* $\mathrm{non}(I_n^*) = \mathfrak{s}$ *and* $\mathrm{non}(S_n) = \aleph_0\text{-}\mathfrak{s}$.

Proof. Suppose that $T \subseteq n^\omega$, $card\,(T) = \mathrm{non}(I_n^*)$ and $T \notin I_n^*$. Let

$$\mathcal{S} = \{x^{-1}[\{i\}] : i \in \{1, \dots, n\} \ \& \ x \in T\} \cap [\omega]^\omega.$$

We claim that for each $A \in [\omega]^\omega$ there exists $S \in \mathcal{S}$ such that $card\,(A \cap S) = card\,(A \setminus S) = \omega$ and so $\mathfrak{s} \leq card\,(\mathcal{S}) = \mathrm{non}(I_n^*)$. Indeed, let $A \in [\omega]^\omega$. Then the set $A^* = \bigcup\{[A \times \{i\}]^* : i \in \{1, \dots, n\}\}$ belongs to the ideal I_n^*, so for some $x \in T$ we have $x \notin A^*$. We consider such $i < \omega$ that $card\,(A \cap x^{-1}[\{i\}]) = \omega$. Note that the relation $card\,(A \setminus x^{-1}[\{i\}]) < \omega$ is impossible, since $x \notin A^*$. Hence $x^{-1}[\{i\}]$ splits the set A. Therefore the inequality $\mathfrak{s} \leq \mathrm{non}(I_n^*)$ holds.
We shall prove now that $\mathrm{non}(I_2^*) \leq \text{fin-}\mathfrak{s}$. Suppose that $\mathcal{S} \subseteq [\omega]^\omega$ is such a family that $card\,(\mathcal{S}) = \text{fin-}\mathfrak{s}$ and for each $\mathcal{A} \in [[\omega]^\omega]^{<\omega}$ there exists $S \in \mathcal{S}$ which splits each element of $\mathcal{A}$. Then one can check that the set

$$Y = \{(S \times \{1\}) \cup ((\omega \setminus S) \times \{0\}) : S \in \mathcal{S}\} \subseteq n^\omega$$

does not belong to I_2^*. Hence $\mathrm{non}(I_n^*) \leq \mathrm{non}(I_2^*) \leq \text{fin-}\mathfrak{s} \leq \mathfrak{s}$.
Similarly, $\mathrm{non}(S_2) = \aleph_0\text{-}\mathfrak{s}$. According to Lemma 2.1, we have

$$\mathrm{non}(S_2) \geq \mathrm{non}(S_3) \geq \dots,$$

and hence it is sufficient to prove that $\mathrm{non}(S_n) \leq \mathrm{non}(S_{n+1})$ for each natural n. This is easily established by a standard technique from partition calculus, temporarily identifying two elements of $n+1$ to obtain a nearly homogeneous set and then un-identifying those two elements to get homogeneity. □

Our last theorem summarizes properties of ideals I_n, I_n^* and S_n for finite n proved in this paper.

Theorem 4.3. *Let* $2 \leq n \leq \omega$. *Then*

1) $\mathrm{add}(I_n) = \mathrm{add}(I_n^*) = \omega < \mathrm{add}(S_n) = \omega_1$;
2) $\mathrm{non}(I_n) = \omega < \mathrm{non}(I_n^*) = \mathfrak{s} \leq \mathrm{non}(S_n) = \aleph_0\text{-}\mathfrak{s}$;
3) $\mathrm{cov}(I_n) = \mathrm{cov}(I_n^*) = \mathrm{cov}(S_n) = \mathfrak{r}$;
4) $\mathrm{cof}(I_n) = \mathrm{cof}(I_n^*) = \mathrm{cof}(S_n) = \mathfrak{c}$.

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