POSITIVE AND NEGATIVE DEFINITE KERNELS ON TREES

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Let $X = (V, E)$ be a *tree* with the set V of vertices and the set E of edges. For any $x \in V$ we will denote by $N(x)$ the *neighbourhood* of x, i.e. the set $\{v \in V : d(v, z) \leq 1\}$. Suppose that for any $x \in V$ we have a fixed positive definite matrix $A(x) = (a(v, x, w))_{v,w \in N(x)}$ such that $a(v, x, v) = 1$ for any $v \in N(x)$. We define the kernel $\phi : V \times V \to \mathbb{C}$ in the following way: if $x, y \in V$ and $[x, y] = \{x_0 = x, x_1, x_2, \ldots, x_n = y\} \subset V$ is the geodesic from x to y, then we put

$$\phi(x, y) = \prod_{i=0}^{n} a(x_{i-1}, x_i, x_{i+1}) \ ,$$

where, by definition, $x_{-1} = x_0 = x$ and $x_{n+1} = x_n = y$. In particular $\phi(x, x) = 1$. Let us also define the additional kernel

$$\beta(x, y) = \prod_{i=0}^{n-1} a(x_{i-1}, x_i, x_{i+1}) \ ,$$

$\beta(x, x) = 1$, which will help us in computations. Note that for any $i \in \{0, 1, 2, \ldots, n\}$ we have

$$\phi(x, y) = \beta(x, x_i) a(x_{i-1}, x_i, x_{i+1}) \overline{\beta(y, x_i)} \ . \tag{1}$$

We are going to prove:

THEOREM. *ϕ is a positive definite kernel on V.*

We start with the following

LEMMA. *For any positive definite matrix $A = (a(i, j))_{i,j \in I}$, for any fixed $i_0 \in I$ and for any finitely supported complex function s on I,*

$$\left| \sum_{i \in I} a(i, i_o) s(i) \right|^2 \leq a(i_o, i_o) \sum_{i,j \in I} a(i, j) s(i) \overline{s(j)} \ .$$

Harmonic Analysis and Discrete Potential Theory, Edited by M.A. Picardello
Plenum Press, New York, 1992

Proof. Let $\langle .,. \rangle$ be the semidefinite positive scalar product given by the matrix A. Then, by the Cauchy – Schwarz inequality,

$$\left| \sum_{i \in I} a(i, i_o) s(i) \right|^2 = |\langle s, \delta_{i_o} \rangle|^2 \leq \langle s, s \rangle \langle \delta_{i_o}, \delta_{i_o} \rangle$$
$$= a(i_o, i_o) \sum_{i,j \in I} a(i,j) s(i) \overline{s(j)} \ . \qquad \square$$

Proof of the Theorem. For any $z, v \in V$ satisfying $d(z,v) \leq 1$ we define the set

$$V(z,v) = \begin{cases} \{z\} & \text{if } v = z \\ \{x \in V : d(x,z) = d(x,v) + 1\} & \text{otherwise.} \end{cases}$$

Then for any $z \in V$ we have the pairwise disjoint decomposition $V = \bigcup_{v \in N(z)} V(z,v)$. Moreover, for any z, z' such that $d(z,z') = 1$ we also have the partition $V(z,z') = \bigcup_{v \in N(z') \setminus \{z\}} V(z',v)$.

We shall prove by induction the following statement: *for any $z \in V$ and for any finitely supported complex function f on V satisfying $f(x) = 0$ when $d(z,x) > n$, we have*

$$\sum_{x,y \in V} \phi(x,y) f(x) \overline{f(y)}$$
$$\geq \sum_{v,w \in N(z)} a(v,z,w) \left(\sum_{x \in V(z,v)} \beta(x,z) f(x) \right) \overline{\left(\sum_{y \in V(z,w)} \beta(y,z) f(y) \right)} \ .$$

For $n = 0$ the statement is obvious. Assume that it is proved for n; we shall prove it for fixed $z \in V$ and for $n+1$. Note that, by (1), if $z \in [x,y]$ then the coefficients of $f(x)\overline{f(y)}$ on the left and on the right hand side are equal. Therefore, by (1), we only need to prove that, for any $z' \in N(z) \setminus \{z\}$:

$$\sum_{z,y \in V(z,z')} \phi(x,y) f(x) \overline{f(y)}$$
$$\geq \left(\sum_{x \in V(z,z')} \beta(x,z) f(x) \right) \overline{\left(\sum_{y \in V(z,z')} \beta(y,z) f(y) \right)} \ ,$$

(recall that $a(z',z,z') = 1$). Let us compute the right hand side of the inequality:

$$\left(\sum_{x \in V(z,z')} \beta(x,z) f(x) \right) \overline{\left(\sum_{y \in V(z,z')} \beta(y,z) f(y) \right)}$$
$$= | \sum_{x \in V(z,z')} \beta(x,z) f(x) |^2$$
$$= | \sum_{v \in N(z') \setminus \{z\}} \sum_{x \in V(z',v)} \beta(x,z) f(x) |^2$$
$$= \left| \sum_{v \in N(z') \setminus \{z\}} a(v,z',z) \left(\sum_{x \in V(z',v)} \beta(x,z') f(x) \right) \right|^2$$
$$= | \sum_{v \in N(z') \setminus \{z\}} a(v,z',z) S(v) |^2$$

where $S(v) = \sum_{x \in V(z',v)} \beta(x,z')f(x)$.

Applying the induction assumption to z' and to $f\chi_{V(z,z')}$ (where $\chi_{V(z,z')}$ is the characteristic function of the set $V(z,z')$) and using the Lemma we get

$$\sum_{x,y \in V(z,z')} \phi(x,y)f(x)\overline{f(y)}$$

$$\geq \sum_{v,w \in N(z')\setminus\{z\}} a(v,z',w)\left(\sum_{x \in V(z',v)} \beta(x,z')f(x)\right)\overline{\left(\sum_{y \in V(z',w)} \beta(y,z')f(y)\right)}$$

$$= \sum_{v,w \in N(z')\setminus\{z\}} a(v,z',w)S(v)\overline{S(w)} \geq |\sum_{v \in N(z')\setminus\{z\}} a(v,z',z)S(v)|^2 \ , \qquad \square$$

which concludes the proof.

Now, suppose that for any $x \in V$ we have a fixed negative definite matrix $C(x) = (c(v,x,w))_{v,w \in N(x)}$ such that $c(v,x,v) = 0$ for any $v \in N(x)$. We define the kernel $\psi : V \times V \to \mathbb{C}$ in the following way: if $x, y \in V$ and $[x,y] = \{x_0 = x, x_1, x_2, \ldots, x_n = y\} \subset V$ is the geodesic from x to y then we put

$$\psi(x,y) = \sum_{i=0}^{n} c(x_{i-1}, x_i, x_{i+1}) \ ,$$

where, as before, $x_{-1} = x_0 = x$ and $x_{n+1} = x_n = y$. In particular $\psi(x,x) = 0$.

COROLLARY 1. *ψ is a negative definite kernel on V.*

Proof. Let t be a fixed positive number and define $\phi_t(x,y) = \exp(-t\psi(x,y))$. Then $\phi_t(x,y) = \prod_{i=0}^{n} a_t(x_{i-1}, x_i, x_{i+1})$, where, for any $x \in V$, the matrix $A_t(x) = (a_t(v,x,w))_{v,w \in N(x)}$ is given by $a_t(v,x,w) = \exp(-tc(v,x,w))$. By Schoenberg's theorem all matrices $A_t(x)$ are positive definite, so ϕ_t is a positive definite kernel for any $t > 0$. Applying Schoenberg's theorem again, we infer that ψ is a negative definite kernel. $\square$

As a corollary we obtain a result of A. Valette (cf. [2]).

COROLLARY 2. *Let f be any real valued function on V satisfying $f(x) \leq \frac{1}{\deg(x)}$ and define a kernel ψ on V by*

$$\psi(x,y) = \begin{cases} 0 & \text{if } x = y \\ d(x,y) - \frac{f(x)+f(y)}{2} & \text{if } x \neq y \ . \end{cases}$$

Then ψ is negative definite on V.

Proof. For any $x \in V$ we define the matrix $C(x) = (c(v,x,w))_{v,w \in N(x)}$ in the following way: $c(v,x,v) = 0$ for $v \in N(x)$, $c(v,x,x) = c(x,x,v) = \frac{1-f(x)}{2}$ for $v \in N(x)\setminus\{x\}$ and $c(v,x,w) = 1$ for $v, w \in N(x)\setminus\{x\}$, $v \neq w$. We shall prove that $C(x)$ is negative definite. By [1, Lemma 3.2.1] it is enough to show that the matrix $B = (b(v,w))_{v,w \in N(x)}$ with $b(v,w) = c(v,x,x) + c(w,x,x) - c(v,x,w)$ is positive definite. We have

$$b(v,w) = \begin{cases} 0 & \text{if } v = x \text{ or } w = x \\ 1 - f(x) & \text{if } v = w \neq x \\ -f(x) & \text{if } v \neq w,\ v \neq x,\ w \neq x \ . \end{cases}$$

By [2, Proposition 1] the matrix B is positive definite, so $C(x)$ is negative definite.

Now, let ψ_C be the negative definite kernel given by the system of matrices $(C(x))_{x \in V}$. Then $\psi_C(x,x) = 0$ and for $x \neq y$ we have

$$\psi_C(x,y) = \frac{1 - f(x)}{2} + d(x,y) - 1 + \frac{1 - f(y)}{2} = \psi(x,y) \ . \qquad \square$$

REFERENCES

[1] Ch. Berg, J. Christensen, P. Ressel, "Harmonic Analysis on Semigroups," Springer-Verlag, 1984.

[2] A. Valette, *Negative definite kernels on trees*, these proceedings.