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Free probability on algebras with infinitely many states

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Abstract. We study noncommutative probability spaces endowed with infinite sequences of states. Following ideas of Cabanal-Duvillard we extend the notion of conditional freeness. Free product of such spaces is justified by constructing an appropriate $*$-representation. Finally, we provide limit theorems and describe the sequences of orthogonal polynomials related to the limit measures.

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1. Introduction

The idea of noncommutative probability is to work with a unital complex $*$-algebra $\mathscr{A}$, elements of which are called *random variables*, equipped with a *state* $\phi : \mathscr{A} \to \mathbf{C}$ (i.e. a linear function satisfying $\phi(1) = 1$ and $\phi(a^*a) \geq 0$ for $a \in \mathscr{A}$), which plays the role of a probability measure. According to Voiculescu [V, VDN], a family $\{\mathscr{A}\}_{i\in I}$ of subalgebras of $\mathscr{A}$ is said to be *free* if $\phi(a_1a_2\cdots a_m) = 0$ holds whenever $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $i_1 \neq i_2 \neq \cdots \neq i_m$ and $\phi(a_1) = \phi(a_2) = \cdots = \phi(a_m) = 0$. He proved that if $\{(\mathscr{A}_i, \phi_i)\}_{i\in I}$ are probability spaces and if $\mathscr{A}$ is the unital free product $*_{i\in I}\mathscr{A}_i$ then there is a unique state ϕ on $\mathscr{A}$ satisfying

1) $\phi|_{\mathscr{A}_i} = \phi_i$ for every $i \in I$,
2) The family $\{\mathscr{A}_i\}_{i\in I}$ is free in $(\mathscr{A}, \phi)$.

Bożejko, Leinert and Speicher [BS, BLS] considered probability spaces $\mathscr{A}$ endowed with a *pair* (ϕ, ψ) of states. A family of subalgebras $\{\mathscr{A}_i\}_{i\in I}$ is

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said to be *conditionally free* if $\phi(a_1a_2\cdots a_m) = \phi(a_1)\phi(a_2)\cdots\phi(a_m)$ and $\psi(a_1a_2\cdots a_m) = 0$ whenever $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}, i_1 \neq i_2 \neq \cdots \neq i_m$ and $\psi(a_1) = \psi(a_2) = \cdots = \psi(a_m) = 0$. Similarly as before, they proved that if $\{(\mathscr{A}_i, \phi_i, \psi_i)\}_{i\in I}$ is a family of such spaces then there is a unique pair (ϕ, ψ) of states on $\mathscr{A} = *_{i\in I}\mathscr{A}_i$ such that

1) $\phi|_{\mathscr{A}_i} = \phi_i$ and $\psi|_{\mathscr{A}_i} = \psi_i$ for every $i \in I$,

2) The family $\{\mathscr{A}_i\}_{i\in I}$ is conditionally free in $(\mathscr{A}, \phi)$.

It was noticed in [M] that the notion of conditional freeness, together with the corresponding combinatorics, can be extended to spaces in which the first state ϕ is an operator-valued one, i.e. is a completely positive map $\phi : \mathscr{A} \to \mathscr{B}(\mathscr{H}_0)$ satisfying $\phi(1) = \text{Id}$.

Next step was made by Cabanal-Duvillard who investigated spaces $\mathscr{A}$ equipped with a sequence $\{\phi_k\}_{k=0}^{\infty}$ of states. In this situation freeness of $\{\mathscr{A}_i\}_{i\in I}$ is defined by imposing that for every $k \geq 0$

$$\phi_k(a_m\cdots a_1a_0) = \phi_k(a_m\cdots a_1)\phi_k(a_0)$$

holds if $a_0 \in \mathscr{A}_{i_0}, a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $i_0 \neq i_1 \neq \cdots \neq i_m$ and $\phi_{k+1}(a_1) = \phi_{k+2}(a_2) = \cdots = \phi_{k+m}(a_m) = 0$. This notion also admits free product but, in spite of the previous cases, this operation is no longer associative. Cabanal-Duvillard proved that in this framework every symmetric compactly supported probability measure can be reached in the central limit theorem.

The aim of this paper is to study such spaces in details and from more general point of view. Namely, for an index set I define $S(I)$ to be the set of formal words $i_1i_2\cdots i_m, m \geq 0$, such that $i_k \in I$ and $i_1 \neq i_2 \neq \cdots \neq i_m$. Let $\{\mathscr{A}_i\}$ be a family of complex unital $*$-algebras and $\mathscr{A} = *_{i\in I}\mathscr{A}_i$. Assume that $\mathscr{H}_0$ is a fixed Hilbert space and that for every $i \in I$ we are given an operator-valued state $\phi_i : \mathscr{A}_i \to \mathscr{B}(\mathscr{H}_0)$ and for every $i_1i_2\cdots i_m \in S(I)$, with $m \geq 2$, we have a state $\phi_{i_1i_2\cdots i_m} : \mathscr{A}_{i_1} \to \mathbf{C}$ of $\mathscr{A}_{i_1}$. Then we define a function $\phi : \mathscr{A} \to \mathscr{B}(\mathscr{H}_0)$ by requiring that

$$\phi(a_1\cdots a_m) = \phi(a_1\cdots a_{m-1})\phi_{i_m}(a_m)$$

holds whenever $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}, i_1 \neq i_2 \neq \cdots \neq i_m$ and

$$\phi_{i_1\cdots i_m}(a_1) = \phi_{i_2\cdots i_m}(a_2) = \cdots = \phi_{i_{m-1}i_m}(a_{m-1}) = 0.$$

In Section 4, Theorem 1, we provide a formula for evaluating $\phi(a_1a_2\cdots a_m)$ for arbitrary $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$ and $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, which in the (conditionally) free case, i.e. when (for $m \geq 2$) $\phi_{i_1\cdots i_m}$ depends only on i_1, is different from that given in [S2, S3] (resp. in [BLS, M]). The formula presented here involves a family $\text{NC}(\mathbf{i})$ of noncrossing partitions related to the sequence $\mathbf{i}$ (see Section 2) and *boolean cumulants*, which are defined and studied in Section 3.

Section 5 contains a proof, by constructing an appropriate representation, that ϕ is an operator-valued state. Next we confine ourselves to probability spaces with a sequence $\{\phi_k\}_{k=0}^{\infty}$ of states, ϕ_0 is allowed to be an operator-valued one. In Section 7 we prove, using the formula from Theorem 1, the limit theorems on such spaces. Finally we describe the sequences of orthogonal polynomials related to the central and Poisson limit theorem.

2. Preliminaries

In the following two sections we provide the combinatorial background which will be needed later on. The notion of noncrossing partition of a linearly ordered set, first studied by Kreweras [K] from a purely combinatorial point of view, was successfully applied by Speicher [S2, S3] to free probability. Here we need to study a class $\mathrm{NC}(\mathbf{i})$ of noncrossing partitions of the set $\{1, 2, \ldots, m\}$ related to a given function $\mathbf{i}$ on this set.

By a *partition* of a set X we mean such a family π of nonempty subsets of X that $\bigcup \pi = X$ and for $V, W \in \pi$ either $V \cap W = \emptyset$ or $V = W$. The corresponding equivalence relation on X will be denoted by $\overset{\pi}{\sim}$ or simply by π, so $p\pi q$ means that $p, q \in V$ for some $V \in \pi$.

Now let $(X, <)$ be a finite linearly ordered set. A partition π of X is called *noncrossing* if $k < p < l < q$, $k, l \in V \in \pi$ and $p, q \in W \in \pi$ implies $V = W$. The class of all noncrossing partitions of X will be denoted $\mathrm{NC}(X)$. A block $V \in \pi \in \mathrm{NC}(X)$ is said to be *inner* if there is another block $W \in \pi$ and elements $p, q \in W$ such that $p < k < q$ for every $k \in V$ (in this situation we write $V \prec_\pi W$). Otherwise V is called *outer*. The family of all inner (resp. outer) blocks of a noncrossing partition π will be denoted by $\pi(i)$ (resp. $\pi(o)$). For $V \in \pi$ we define the *depth* of V by $d(V, \pi) = d(V) := |\{W \in \pi : V \prec_\pi W\}|$ (in particular $d(V, \pi) = 0$ iff $V \in \pi(o)$) and if $V \in \pi(i)$ then we define the *successor* $V'(\pi) = V'$ of V as the smallest, with respect to $\prec_\pi$, block in $\{W \in \pi : V \prec_\pi W\}$. A noncrossing partition π is called *boolean* if $\pi(i) = \emptyset$ (which means that every block is an interval), and the class of boolean partition of X we denote $\mathrm{Bo}(X)$. For $X = \{1, \ldots, m\}$ we write $\mathrm{NC}(m)$ and $\mathrm{Bo}(m)$ for $\mathrm{NC}(X)$ and $\mathrm{Bo}(X)$, respectively. Having a noncrossing partition π of X and a product $\prod_{V\in\pi} f(V)$ we will assume that the factors corresponding to the outer blocks of π are in the same order as these blocks.

For a function $\mathbf{i} : X \to I$ we define $\mathrm{NC}(\mathbf{i})$ to be the set of all partitions π of X satisfying the following three conditions:

1) π is noncrossing.
2) If $k, l \in V \in \pi$ then $\mathbf{i}(k) = \mathbf{i}(l)$. We define $\mathbf{i}(V) := \mathbf{i}(k)$.
3) If $V \in \pi(i)$ then $\mathbf{i}(V) \neq \mathbf{i}(V')$.

We define *type of* V as the formal word $t(V, \pi) = t(V) := \mathbf{i}\ (V)\ \mathbf{i}(V') \cdots \mathbf{i}(V^{(d)})$, where $d = d(V, \pi)$.

Assume that X° is a finite linearly ordered set, $x \in X^\circ$ and x° is the successor of x in X°. For a partition σ of X° we define a partition $\sigma|(x = x^\circ)$ of $X = X^\circ \backslash \{x^\circ\}$ by "gluing" x with x° i.e.

$$\sigma|(x = x^\circ) := (\sigma \backslash \{W, W^\circ\}) \cup \{(W \cup W^\circ) \backslash \{x^\circ\}\}$$

if $x \in W \in \sigma$, $x^\circ \in W^\circ \in \sigma$, possibly $W = W^\circ$. Note that if $\mathbf{i}^\circ : X^\circ \to I$ with $\mathbf{i}^\circ(x) = \mathbf{i}^\circ(x^\circ)$ and $\sigma \in \mathrm{NC}(\mathbf{i}^\circ)$ then $\sigma|(x = x^\circ) \in \mathrm{NC}(\mathbf{i})$, where $\mathbf{i}$ is the restriction of $\mathbf{i}^\circ$ to X. On the other hand, if $\pi \in \mathrm{NC}(\mathbf{i})$ is a partition of X then there are exactly two partitions $\sigma_k \in \mathrm{NC}(\mathbf{i}^\circ)$ of X° satisfying $\sigma_k|(x = x^\circ) = \pi$, namely if $x \in V \in \pi$ then

$$\sigma_1 = (\pi \backslash \{V\}) \cup \{V \cup \{x^\circ\}\}$$

and

$$\sigma_2 = (\pi \backslash \{V\}) \cup \{\{w \in V : w \leq x\}, \{w \in V : x < w\} \cup \{x^\circ\}\}.$$

For a sequence $\mathbf{a} = (a_1, \ldots, a_m)$ and a set of integers $V = \{k_1, \ldots, k_s\}$, with $1 \leq k_1 < k_2 < \cdots < k_s \leq m$, we define a subsequence $V(\mathbf{a}) = V(a_1, \ldots, a_m) := (a_{k_1}, a_{k_2}, \ldots, a_{k_s})$ and a product $\Pi V(\mathbf{a}) := a_{k_1} a_{k_2} \cdots a_{k_m}$.

3. Boolean cumulant

For unital algebras $\mathscr{A}, \mathscr{B}$ over a field K and for a linear function $\Phi : \mathscr{A} \to \mathscr{B}$ satisfying $\Phi(1) = 1$ we define *boolean cumulant* of Φ as a function

$$R : \bigcup_{m=1}^{\infty} \underbrace{\mathscr{A} \times \cdots \times \mathscr{A}}_{m \text{ times}} \to \mathscr{B},$$

satisfying the following recurrence

$$\Phi(a_1 \cdots a_m) = \sum_{\pi \in \mathrm{Bo}(m)} \prod_{V \in \pi} R(V(a_1, \ldots, a_m)),$$

for every $m \geq 1$, $a_1, \ldots, a_m \in \mathscr{A}$. Equivalently (see [SW]),

$$R(a_1, \ldots, a_m) = \sum_{\pi \in \mathrm{Bo}(m)} (-1)^{|\pi|-1} \prod_{V \in \pi} \Phi(\Pi V(a_1, \ldots, a_m)).$$

In particular, R restricted to $\mathscr{A} \times \cdots \times \mathscr{A}$, m times, is m-linear.

Boolean cumulants were introduced by Speicher and Woroudi [SW] in order to define a new convolution of probability measures on the real line. Here we will need the following

Lemma 1. *For arbitrary $m \geq 1$, $a_1, \ldots, a_m, a'_p \in \mathscr{A}$, $1 \leq p \leq m$, we have*

$$\begin{aligned} &R(a_1, \ldots, a_{p-1}, a_p a'_p, a_{p+1}, \ldots, a_m) \\ &= R(a_1, \ldots, a_{p-1}, a_p, a'_p, a_{p+1}, \ldots, a_m) \\ &\quad + R(a_1, \ldots, a_p) R(a'_p, a_{p+1}, \ldots, a_m). \end{aligned}$$

Proof. Putting $X' = \{1, \ldots, p, p', p+1, \ldots, m\}$ we have

$$\begin{aligned} &R(a_1, \ldots, a_p a'_p, \ldots, a_m) \\ &= \sum_{\pi \in \mathrm{Bo}(m)} (-1)^{|\pi|-1} \prod_{V \in \pi} \Phi(\Pi V(a_1, \ldots, a_p a'_p, \ldots, a_m)) \\ &= \sum_{\substack{\sigma \in \mathrm{Bo}(X') \\ p \sigma p'}} (-1)^{|\sigma|-1} \prod_{B \in \sigma} \Phi(\Pi B(a_1, \ldots, a_p, a'_p, \ldots, a_m)) \\ &= R(a_1, \ldots, a_p, a'_p, \ldots, a_m) \\ &\quad - \sum_{\substack{\sigma \in \mathrm{Bo}(X') \\ p \not\sigma p'}} (-1)^{|\sigma|-1} \prod_{B \in \sigma} \Phi(\Pi B(a_1, \ldots, a_p, a'_p, \ldots, a_m)) \\ &= R(a_1, \ldots, a_p, a'_p, \ldots, a_m) + \sum_{\substack{\sigma_1 \in \mathrm{Bo}(1, \ldots, p) \\ \sigma_2 \in \mathrm{Bo}(p', p+1, \ldots, m)}} (-1)^{|\sigma_1|+|\sigma_2|-2} \\ &\quad \times \prod_{B \in \sigma_1} \Phi(\Pi B(a_1, \ldots, a_p)) \prod_{B \in \sigma_2} (\Pi B(a'_p, a_{p+1}, \ldots, a_m)) \\ &= R(a_1, \ldots, a_p, a'_p, \ldots, a_m) + R(a_1, \ldots, a_p) R(a'_p, a_{p+1}, \ldots, a_m). \end{aligned}$$

□

Corollary 1. *Assume that $m \geq 2$, $a_1, \ldots, a_m \in \mathscr{A}$.*
a) If $a_1 = 1$ or $a_m = 1$ then $R(a_1, \ldots, a_m) = 0$.
b) If $a_p = 1$ for some $1 < p < m$ then

$$R(a_1, \ldots, a_{p-1}, 1, a_{p+1}, \ldots, a_m) = R(a_1, \ldots, a_{p-1}, a_{p+1}, \ldots, a_m).$$

Proof. To prove a) we only need to put $p = 1$ and $a_1 = 1$ or $p' = m$ and $a'_m = 1$. Applying a) to the lemma we get b). □

4. General free product

Fix an index set I and define a set of formal words

$$S(I) = \{i_1 i_2 \cdots i_m : m \geq 0, i_k \in I \text{ and } i_1 \neq i_2 \neq \cdots \neq i_m\}.$$

The empty word in $S(I)$ will be denoted by e. Let $\{\mathscr{A}_i\}_{i \in I}$, $\mathscr{B}$ be unital algebras over a field K and assume that for every $i \in I$ we have a linear function $\phi_i : \mathscr{A}_i \to \mathscr{B}$, with $\phi_i(1) = 1$, and for every $u = i_1 \cdots i_m \in S(I)$ such that $m \geq 2$, $i_1 = i$, a linear function $\phi_u : \mathscr{A}_i \to K$, with $\phi_u(1) = 1$. By R_u we will denote the boolean cumulant function

$$R_u : \bigcup_{m=1}^{\infty} \underbrace{\mathscr{A}_i \times \cdots \times \mathscr{A}_i}_{m \text{ times}} \to \begin{cases} \mathscr{B} & \text{if } |u| = 1; \\ K & \text{if } |u| > 1, \end{cases}$$

of ϕ_u, where $|u|$ denotes the length of the word $u \in S(I)$. Now we define a linear function $\phi = *_{u \in S(I)} \phi_u$ on the unital free product $\mathscr{A} = *_{i \in I} \mathscr{A}_i$ by putting $\phi(1) = 1$ and

$$\phi(a_1 \cdots a_m) = \phi(a_1 \cdots a_{m-1}) \phi_{i_m}(a_m)$$

whenever $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $i_1 \neq i_2 \neq \cdots \neq i_m$ and

$$\phi_{i_1 i_2 \cdots i_m}(a_1) = \phi_{i_2 \cdots i_m}(a_2) = \cdots = \phi_{i_{m-1} i_m}(a_{m-1}) = 0.$$

It is clear that this defines ϕ uniquely on $\mathscr{A}$. Note that special case of this product, when $\mathscr{B} = K = \mathbf{C}$ and when ϕ_u, $u = i_1 \cdots i_m \in S(I)$, depends only on i_1 and on $h(u)$, where h ("hauteur") is a function $S(I) \to \mathbf{N}$, was studied by Cabanal-Duvillard, see [CD, CDI].

Theorem 1. *For arbitrary* $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, *we have*

$$\phi(a_1 \cdots a_m) = \sum_{\pi \in \mathrm{NC}(\mathbf{i})} \prod_{V \in \pi} R_{t(V)}(V(a_1, \ldots, a_m)) .$$

Proof. Denote the right hand side by $\Phi(a_1, \ldots, a_m)$. First we show that if $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $a'_p \in \mathscr{A}_{i_p}$ then

$$\Phi(a_1, \ldots, a_p, a'_p, \ldots, a_m) = \Phi(a_1, \ldots, a_p a'_p, \ldots, a_m) .$$

Denote $\mathbf{i}' = (i_1, \ldots, i_{p-1}, i_p, i_p, i_{p+1}, \ldots, i_m)$, $X' = \{1, \ldots, p-1, p, p', p+1, \ldots, m\}$. Fix $\pi \in \mathrm{NC}(\mathbf{i})$ and take both $\sigma_1, \sigma_2 \in \mathrm{NC}(\mathbf{i}')$ satisfying $\sigma_k|(p = p') = \pi$. If $p \in V \in \pi$ then, putting $W = V \cup \{p'\}$, W_-

$= \{w \in V : w \leq p\}$, $W_+ = \{w \in V : p < w\} \cup \{p'\}$, we have $\sigma_1 = (\pi \setminus \{V\}) \cup \{W\}$, $\sigma_2 = (\pi \setminus \{V\}) \cup \{W_-, W_+\}$. Note that by Lemma 1

$$\begin{aligned}
&R_u(V(a_1, \ldots, a_p a'_p, \ldots, a_m)) \\
&\quad = R_u(W(a_1, \ldots, a_p, a'_p, \ldots, a_m)) + R_u(W_-(a_1, \ldots, a_p, a'_p, \ldots, a_m)) \\
&\qquad \times R_u(W_+(a_1, \ldots, a_p, a'_p, \ldots, a_m)).
\end{aligned}$$

Therefore

$$\begin{aligned}
&\Phi(a_1, \ldots, a_p a'_p, \ldots, a_m) \\
&\quad = \sum_{\pi \in \mathrm{NC}(\mathbf{i})} \prod_{V \in \pi} R_{t(V)}(V(a_1, \ldots, a_p a'_p, \ldots, a_m)) \\
&\quad = \sum_{\pi \in \mathrm{NC}(\mathbf{i})} \sum_{\substack{\sigma \in \mathrm{NC}(\mathbf{i}') \\ \sigma|(p=p')=\pi}} \prod_{W \in \sigma} R_{t(W)}(W(a_1, \ldots, a_p, a'_p, \ldots, a_m)) \\
&\quad = \Phi(a_1, \ldots, a_p, a'_p, \ldots, a_m)
\end{aligned}$$

and hence $\Phi(a_1, \ldots, a_m)$ depends only on the product $a_1 \cdots a_m$.

It remains to show that if $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $i_1 \neq i_2 \neq \cdots \neq i_m$ and $\phi_{i_1 i_2 \cdots i_m}(a_1) = \cdots = \phi_{i_{m-1} i_m}(a_{m-1}) = 0$ then $\Phi(a_1, \ldots, a_m) = \Phi(a_1, \ldots, a_{m-1}) \phi_{i_m}(a_m)$.

Assume that $\{m\}$ is not a block in π. Then we show that there is a one-element block $\{k\}$ of π satisfying $R_{t(\{k\})}(a_k) = 0$. Note that by the assumption that $i_1 \neq i_2 \neq \cdots \neq i_m$ one element blocks in π do exist, namely, every minimal, with respect to "$\prec_\pi$", block must be of this sort. Let k be the last number satisfying $\{k\} \in \pi$. We claim that $t(\{k\}) = i_k i_{k+1} \cdots i_m$. It holds because if $k < l \leq m$, $l \in V \in \pi$ then $V \cap \{k+1, k+2, \ldots, m\} = \{l\}$. Indeed, if $k < l < l' \leq m$ and $l, l' \in V$ then $l' \neq l+1$, as $i_l \neq i_{l+1}$, and in the interval $\{l+1, \ldots, l'-1\}$ we would have a minimal block, necessarily one-element one, later than $\{k\}$, which is a contradiction. Consequently, every $l \in \{k+1, \ldots, m\}$ belongs to a block $V_l \in \pi$ and $\{k\} \prec_\pi V_{k+1} \prec_\pi V_{k+2} \prec_\pi \cdots \prec_\pi V_m$. Hence $t(\{k\}) = i_k i_{k+1} \cdots i_m$ and $R_{t(\{k\})}(a_k) = \phi_{i_k \cdots i_m}(a_k) = 0$. Therefore, putting $\mathbf{i}_0 = (i_1, \ldots, i_{m-1})$, we obtain

$$\begin{aligned}
\Phi(a_1, \ldots, a_m) &= \sum_{\pi \in \mathrm{NC}(\mathbf{i})} \prod_{V \in \pi} R_{t(V)}(V(a_1, \ldots, a_m)) \\
&= \sum_{\substack{\pi \in \mathrm{NC}(\mathbf{i}) \\ \{m\} \in \pi}} \prod_{V \in \pi} R_{t(V)}(V(a_1, \ldots, a_m))
\end{aligned}$$

$$= \sum_{\sigma \in \mathrm{NC}(\mathbf{i}_0)} \prod_{W \in \sigma} R_{t(W)}(W(a_1, \ldots, a_{m-1}))\phi_{i_m}(a_m)$$
$$= \Phi(a_1, \ldots, a_{m-1})\phi_{i_m}(a_m),$$

which completes the proof.

□

Remarks.

1. One can see that the number k choosen in the proof satisfies $k \geq \left[\frac{m+2}{2}\right]$, where $[x]$ denotes the entire part of a real number x, so in the definition of $\phi = *_{i \in S(I)} \phi_u$ we only need to assume that $\phi_{i_k i_{k+1} \cdots i_m}(a_k) = 0$ holds for $\left[\frac{m+2}{2}\right] \leq k < m$.

2. Note the alternative definition of $\phi = *_{i \in S(I)} \phi_u$, namely: if $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}, i_1 \neq i_2 \neq \cdots \neq i_m$ and $\phi_{i_k \cdots i_2 i_1}(a_k) = 0$ for $1 < k \leq m$ (or merely for $1 < k \leq \left[\frac{m+1}{2}\right]$) then $\phi(a_1 a_2 \ldots a_m) = \phi_{i_1}(a_1)\phi(a_2 \cdots a_m)$. To see that this is equivalent we need to modify the proof choosing k as the *first*, instead of the last, number satisfying $\{k\} \in \pi$.

3. In the case when $\phi_{i_1 \cdots i_m} = \phi_{i_1 i_2} := \psi_{i_1}$ for every $i_1 \cdots i_m \in S(I)$, with $m \geq 2$ (the conditionally free product or the free product if $\mathscr{B} = K$ and $\psi_i = \phi_i$ for every $i \in I$), also another formula holds (see [S2, S3, BLS, M]), namely if $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$ then

$$\phi(a_1 \cdots a_m) = \sum_{\pi \in \mathrm{NC}'(\mathbf{i})} \prod_{V \in \pi(i)} r'_{\mathbf{i}(V)}(V(a_1, \ldots, a_m)) \times \prod_{V \in \pi(o)} R'_{\mathbf{i}(V)}(V(a_1, \ldots, a_m)),$$

where $\mathrm{NC}'(\mathbf{i})$ denotes the class of noncrossing partitions π of $\{1, \ldots, m\}$ satisfying $k, l \in V \in \pi \Rightarrow i_k = i_l (:= \mathbf{i}(V))$, and where the cumulant functions

$$R'_i : \bigcup_{m=1}^{\infty} \underbrace{\mathscr{A}_i \times \cdots \times \mathscr{A}_i}_{m \text{ times}} \to \mathscr{B}; \qquad r'_i : \bigcup_{m=1}^{\infty} \underbrace{\mathscr{A}_i \times \cdots \times \mathscr{A}_i}_{m \text{ times}} \to K$$

are defined by the following recurrence:

$$\psi_i(b_1 \cdots b_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi} r'_i(V(b_1, \ldots, b_n)),$$

$$\phi_i(b_1 \cdots b_n) = \sum_{\pi \in \mathrm{NC}(n)} \prod_{V \in \pi(i)} r'_i(V(b_1, \ldots, b_n)) \prod_{V \in \pi(o)} R'_i(V(b_1, \ldots, b_n)),$$

for every $b_1, \ldots, b_n \in \mathscr{A}_i$.

Corollary 2. *If* $a_1 \in \mathscr{A}_{i_1}, \dots, a_m \in \mathscr{A}_{i_m}$ *and* $\{i_1, \dots, i_p\} \cap \{i_{p+1}, \dots, i_m\} = \emptyset$ *for some* $1 \leq p < m$ *then*

$$\phi(a_1 \cdots a_m) = \phi(a_1 \cdots a_p)\phi(a_{p+1} \cdots a_m).$$

Proof. It holds because every $\pi \in \mathrm{NC}(i_1, \dots, i_m)$ is a product $\pi = \pi_1\pi_2$, $\pi_1 \in \mathrm{NC}(i_1, \dots, i_p)$, $\pi_2 \in \mathrm{NC}(i_{p+1}, \dots, i_m)$. □

5. Positivity

In this part we prove that positivity of all ϕ_u's implies that of $\phi = *_{u \in S(I)}\phi_u$. For this purpose we generalise the construction of the free product of representations given by Voiculescu [V, VDN] (see also [BS, M, CD]).

Let $\{\mathscr{A}_i\}_{i \in I}$ be a family of unital complex $*$-algebras. We assume that $\mathscr{H}_0$ is a fixed Hilbert space and that for every $i \in I$ we have a $*$-representation $\pi_i : \mathscr{A}_i \to \mathscr{B}(\mathscr{H}_0 \oplus \mathscr{K}_i)$. By ϕ_i we denote the corresponding *operator valued state*, i.e. for $a \in \mathscr{A}_i$ we put $\phi_i(a) = P_i\pi_i(a)|_{\mathscr{H}_0}$, where P_i is the orthogonal projection of $\mathscr{H}_0 \oplus \mathscr{K}_i$ onto $\mathscr{H}_0$. Moreover, we assume that for every $u = i_1 \cdots i_m \in S(I)$, with $m \geq 2$, we are given a $*$-representation $\pi_u : \mathscr{A}_{i_1} \to \mathscr{B}(\mathbf{C}\xi_u \oplus \mathscr{K}_u)$, where ξ_u is a unit vector. Let ϕ_u denote the corresponding *state* on $\mathscr{A}_{i_1}$, i.e. $\phi_u(a) = \langle \pi_u(a)\xi_u, \xi_u \rangle$. In particular all the functions ϕ_u are completely positive.

Now we are going to define a $*$-representation of the unital free product $\mathscr{A} = *_{i \in I}\mathscr{A}_i$. For $u = i_1 \dots i_m \in S(I)$, with $m \geq 1$, we put

$$\mathscr{H}_{i_1 i_2 \cdots i_m} = \mathscr{K}_{i_1 i_2 \cdots i_m} \otimes \mathscr{K}_{i_2 \cdots i_m} \otimes \cdots \otimes \mathscr{K}_{i_m}.$$

In particular we have

$$\mathscr{K}_{iu} \otimes \mathscr{H}_u = \mathscr{H}_{iu},$$

whenever $I \ni i \neq i_1$. Define

$$\mathscr{H} = \bigoplus_{u \in S(I)} \mathscr{H}_u.$$

For every $i \in I$ we have the following decomposition

$$\mathscr{H} = (\mathscr{H}_0 \oplus \mathscr{K}_i) \oplus \bigoplus_{\substack{u = i_1 \cdots i_m \in S(I) \\ m \geq 1, i_1 \neq i}} (\mathbf{C}\xi_{iu} \oplus \mathscr{K}_{iu}) \otimes \mathscr{H}_u.$$

Using this decomposition we are able to define on $\mathscr{H}$ a $*$-representation $\tilde{\pi}_i$ of $\mathscr{A}_i$ acting on $\mathscr{H}_0 \oplus \mathscr{K}_i$ by π_i and on $(\mathbf{C}\xi_{iu} \oplus \mathscr{K}_{iu}) \otimes \mathscr{H}_u$ by $\pi_{iu} \otimes \mathrm{Id}_{\mathscr{H}_u}$. Having $\tilde{\pi}_i$ defined for every $i \in I$ we can define a representation π of $\mathscr{A}$ putting $\pi(a_1 \cdots a_m) = \tilde{\pi}_{i_1}(a_1) \cdots \tilde{\pi}_{i_m}(a_m)$ whenever $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, $i_1 \neq i_2 \neq \cdots \neq i_m$. Denote by ϕ the corresponding state on $\mathscr{A}$, i.e. $\phi(a) = P_0 \pi(a)|_{\mathscr{H}_0}$ for $a \in \mathscr{A}$. Next theorem says that $\phi = *_{u \in S(I)} \phi_u$.

Theorem 2. *Assume that* $a_1 \in \mathscr{A}_{i_1}, a_2 \in \mathscr{A}_{i_2}, \ldots, a_m \in \mathscr{A}_{i_m}$, $m \geq 2$, $i_1 \neq i_2 \neq \cdots \neq i_m$ *and*

$$\phi_{i_1 i_2 \cdots i_m}(a_1) = \phi_{i_2 \cdots i_m}(a_2) = \cdots = \phi_{i_{m-1} i_m}(a_{m-1}) = 0.$$

Then

$$\phi(a_1 a_2 \cdots a_m) = \phi(a_1 a_2 \cdots a_{m-1}) \phi_{i_m}(a_m).$$

Proof. Throughout the proof $S(I)$ will be regarded as a group $(S(I), \circ, e)$ isomorphic to the free product $*_{i \in I} \mathbf{Z}_2$, with I as the set of generators ($i \circ i = e$ for $i \in I$) and with the empty word e as the neutral element.

For $i \in I$, $a \in \mathscr{A}_i$ we decompose $\tilde{\pi}_i(a)$ into a sum $\pi_i^0(a) + \pi_i^1(a)$ in the following way. Take a tensor $x = x_1 \otimes \ldots \otimes x_n$ of *type* $v = j_1 j_2 \ldots j_n \in S(I)$, i.e. $x_1 \in \mathscr{K}_{j_1 j_2 \cdots j_n}, x_2 \in \mathscr{K}_{j_2 \cdots j_n}, \ldots, x_n \in \mathscr{K}_{j_n}$. We have to consider four cases.

1) If $v = e$ then $x \in \mathscr{H}_0$ and we have

$$\begin{aligned} \tilde{\pi}_i(a)x &= P_i \pi_i(a)x + (\mathrm{Id} - P_i)\pi_i(a)x \\ &:= \pi_i^0(a)x + \pi_i^1(a)x. \end{aligned}$$

2) If $v = i$ then $x \in \mathscr{K}_i$ and we have

$$\begin{aligned} \tilde{\pi}_i(a)x &= (\mathrm{Id} - P_i)\pi_i(a)x + P_i \pi_i(a)x \\ &:= \pi_i^0(a)x + \pi_i^1(a)x. \end{aligned}$$

3) Assume that $n \geq 2$ and $j_1 = i$. Then

$$\begin{aligned} \tilde{\pi}_i(a)x &= (\pi_v(a)x_1) \otimes x_2 \otimes \cdots \otimes x_n \\ &= [(\mathrm{Id} - P_v)\pi_v(a)x_1] \otimes x_2 \otimes \cdots \otimes x_n + \langle \pi_v(a)x_1, \xi_v \rangle x_2 \otimes \cdots \otimes x_n \\ &:= \pi_i^0(a)x + \pi_i^1(a)x, \end{aligned}$$

where P_v denotes the orthogonal projection of $\mathbf{C}\xi_v \oplus \mathscr{K}_v$ onto $\mathbf{C}\xi_v$.

4) Finally, consider the case when $n \geq 1$ and $j_1 \neq i$. In this case we have

$$\tilde{\pi}_i(a)x = \pi_{iv}(a)\xi_{iv} \otimes x \;=\; \phi_{iv}(a)x + (\mathrm{Id} - P_{iv})\pi_{iv}(a)\xi_{iv} \otimes x$$
$$:= \pi_i^0(a)x + \pi_i^1(a)x.$$

In this way we have decomposition $\tilde{\pi}_i(a) = \pi_i^0(a) + \pi_i^1(a)$ and $\pi_i^\epsilon(a)$ maps a tensor of type v into a tensor of type $i^\epsilon \circ v$. Therefore if $i_1, \ldots, i_m \in I$, $\epsilon_1, \ldots, \epsilon_m \in \{0, 1\}$, $a_1 \in \mathscr{A}_{i_1}, \ldots, a_m \in \mathscr{A}_{i_m}$, then $\pi_{i_1}^{\epsilon_1}(a_1) \ldots \pi_{i_m}^{\epsilon_m}(a_m)$ maps a tensor of type v into a tensor of type $i_1^{\epsilon_1} \circ \cdots \circ i_m^{\epsilon_m} \circ v$. For a fixed $\xi \in \mathscr{H}_0$ we have

$$\pi(a_1 \cdots a_m)\xi = \sum_{\epsilon_1,\cdots,\epsilon_m \in \{0,1\}} \pi_{i_1}^{\epsilon_1}(a_1) \ldots \pi_{i_m}^{\epsilon_m}(a_m)\xi$$

and

$$P_0\pi(a_1 \cdots a_m)\xi = \sum_{\substack{\epsilon_1,\ldots,\epsilon_m \in \{0,1\} \\ i_1^{\epsilon_1} \circ \ldots \circ i_m^{\epsilon_m} = e}} \pi_{i_1}^{\epsilon_1}(a_1) \cdots \pi_{i_m}^{\epsilon_m}(a_m)\xi.$$

Now assume that $i_1 \neq i_2 \neq \cdots \neq i_m$ and that $\phi_{i_1 i_2 \cdots i_m}(a_1) = \phi_{i_2 \cdots i_m}(a_2) = \cdots = \phi_{i_{m-1} i_m}(a_{m-1}) = 0$. If $i_1^{\epsilon_1} \circ \cdots \circ i_m^{\epsilon_m} = e$ and $\epsilon_m = 1$ then for some $1 < k < m$ we have $\epsilon_k = 0$, $\epsilon_{k+1} = \cdots = \epsilon_m = 1$. In this case

$$x = \pi_{i_{k+1}}^1(a_{k+1}) \cdots \pi_{i_m}^1(a_m)\xi$$

is a tensor of type $i_{k+1} \cdots i_m$ and, by (4), $\pi_{i_k}^0(a_k)x = \phi_{i_k i_{k+1} \cdots i_m}(a_k)x = 0$. Therefore we can confine ourselves to summands with $\epsilon_m = 0$:

$$P_0\pi(a_1 \cdots a_m)\xi = \sum_{\substack{\epsilon_1,\ldots,\epsilon_{m-1} \in \{0,1\} \\ i_1^{\epsilon_1} \circ \cdots \circ i_{m-1}^{\epsilon_{m-1}} = e}} \pi_{i_1}^{\epsilon_1}(a_1) \cdots \pi_{i_{m-1}}^{\epsilon_{m-1}}(a_{m-1})\pi_{i_m}^0(a_m)\xi$$
$$= \phi(a_1 \cdots a_{m-1})\phi_{i_m}(a_m)\xi,$$

which concludes the proof. □

6. Algebraic $\mathscr{B}$-probability spaces

Let $\mathscr{B}$ be a fixed unital algebra over a field K. We will call a pair $(\mathscr{A}, \{\phi_k\}_{k=0}^\infty)$ an *algebraic $\mathscr{B}$-probability space* if $\mathscr{A}$ is a unital algebra over K, ϕ_k

are linear functions, $\phi_0 : \mathscr{A} \to \mathscr{B}$ and $\phi_k : \mathscr{A} \to K$ for $k \geq 1$, satisfying $\phi_k(1) = 1$. Having a family $(\mathscr{A}_i, \{\phi_{i,k}\}_{k=0}^{\infty})$, $i \in I$, of algebraic $\mathscr{B}$-probability spaces we define their *free product* $(\mathscr{A}, \{\phi_k\}_{k=0}^{\infty}) = *_{i\in I}(\mathscr{A}_i, \{\phi_{i,k}\}_{k=0}^{\infty})$ putting $\mathscr{A} = *_{i\in I}\mathscr{A}_i$ and $\phi_k = *_{u\in S(I)}\phi_{u,k}$, where $\phi_{u,k} := \phi_{i,m+k-1}$ for $e \neq u = ii_1i_2\cdots i_m \in S(I)$. It means that for every $k \geq 0$

$$\phi_k(a_m \cdots a_1a_0) = \phi_k(a_m \cdots a_1)\phi_{i_0,k}(a_0)$$

whenever $m \geq 1$, $a_m \in \mathscr{A}_{i_m}, \ldots, a_1 \in \mathscr{A}_{i_1}, a_0 \in \mathscr{A}_{i_0}$, $i_m \neq \cdots \neq i_1 \neq i_0$ and $\phi_{i_m,k+m}(a_m) = \cdots = \phi_{i_2,k+2}(a_2) = \phi_{i_1,k+1}(a_1) = 0$.

Notice that in spite of the free and conditionally free product, this operation is not associative:

Example. Take three algebraic $\mathscr{B}$-probability spaces and their possible free products:

$$(\mathscr{A}, \{\phi_k\}_{k=0}^{\infty}) = (\mathscr{A}_h, \{\phi_{h,k}\}_{k=0}^{\infty}) * (\mathscr{A}_i, \{\phi_{i,k}\}_{k=0}^{\infty}) * (\mathscr{A}_j, \{\phi_{j,k}\}_{k=0}^{\infty}),$$

$$(\mathscr{A}, \{\phi'_k\}_{k=0}^{\infty}) = \Big((\mathscr{A}_h, \{\phi_{h,k}\}_{k=0}^{\infty}) * (\mathscr{A}_i, \{\phi_{i,k}\}_{k=0}^{\infty})\Big) * (\mathscr{A}_j, \{\phi_{j,k}\}_{k=0}^{\infty}),$$

$$(\mathscr{A}, \{\phi''_k\}_{k=0}^{\infty}) = (\mathscr{A}_h, \{\phi_{h,k}\}_{k=0}^{\infty}) * \Big((\mathscr{A}_i, \{\phi_{i,k}\}_{k=0}^{\infty}) * (\mathscr{A}_j, \{\phi_{j,k}\}_{k=0}^{\infty})\Big),$$

where $\mathscr{A} = \mathscr{A}_h * \mathscr{A}_i * \mathscr{A}_j$, and take $a_1, a_2 \in \mathscr{A}_h$, $b_1, b_2 \in \mathscr{A}_i$, $c \in \mathscr{A}_j$. Then, omitting h, i, j, k on the right hand side and writting ϕ_1 and ϕ_2 instead of ϕ_{k+1} and ϕ_{k+2} respectively, we have

$$\begin{aligned}\phi_k(a_1b_1cb_2a_2) = \phi''_k(a_1b_1cb_2a_2) &= \phi(a_1)\phi(b_1)\phi(c)\phi(b_2)\phi(a_2)\\ &\quad+ \phi_2(c)\phi_1(b_1b_2)\big[\phi(a_1a_2) - \phi(a_1)\phi(a_2)\big]\\ &\quad+ \phi_1(c)\phi(a_1)\big[\phi(b_1b_2) - \phi(b_1)\phi(b_2)\big]\phi(a_2)\\ &\quad+ \big[\phi_1(c) - \phi_2(c)\big]\phi_1(b_1)\phi_1(b_2)\left[\phi(a_1a_2) - \phi(a_1)\phi(a_2)\right],\end{aligned}$$

while

$$\begin{aligned}\phi'_k(a_1b_1cb_2a_2) = \phi(a_1)\phi(b_1)\phi(c)\phi(b_2)\phi(a_2)&\\ + \phi_1(c)\phi_1(b_1b_2)\big[\phi(a_1a_2) - \phi(a_1)\phi(a_2)\big]&\\ + \phi_1(c)\phi(a_1)\big[\phi(b_1b_2) - \phi(b_1)\phi(b_2)\big]\phi(a_2),&\end{aligned}$$

so the functions ϕ_k, ϕ'_k and ϕ''_k are different.

In this context the following definition seems natural.

Definition. A family $\{\mathscr{A}_i\}_{i\in I}$ of unital subalgebras in an algebraic $\mathscr{B}$-probability space $(\mathscr{A}, \{\phi_k\}_{k=0}^{\infty})$ is said to be *free* if for every $k \geq 0$

$$\phi_k(a_m \cdots a_1 a_0) = \phi_k(a_m \cdots a_1)\phi_k(a_0)$$

whenever $m \geq 1$, $a_m \in \mathscr{A}_{i_m}, \ldots, a_1 \in \mathscr{A}_{i_1}, a_0 \in \mathscr{A}_{i_0}$, $i_m \neq \cdots \neq i_1 \neq i_0$ and $\phi_{k+m}(a_m) = \cdots = \phi_{k+2}(a_2) = \phi_{k+1}(a_1) = 0$.

7. Limit theorems

In this section we assume that $\mathscr{B}$ is a unital complex algebra with a norm $\|\cdot\|$ and that $(\mathscr{A}, \{\phi_k\}_{k=0}^{\infty})$ is an algebraic $\mathscr{B}$-probability space, R_k will denote the boolean cumulant of ϕ_k. We are going to study limit theorems on $(\hat{\mathscr{A}}, \{\hat{\phi}_k\}_{k=0}^{\infty}) = *_{i\in\mathbf{N}}(\mathscr{A}, \{\phi_k\}_{k=0}^{\infty})$. For $a \in \mathscr{A}$, $i \in \mathbf{N}$ we denote by (a, i) the embedding of a into the i-th factor $\mathscr{A}$ of $\hat{\mathscr{A}}$. Note that the next theorem and Corollary 3 were stated in [CD].

Theorem 3 (general limit theorem). *Let $m \geq 1$ be a fixed integer and let for every $N \in \mathbf{N}$ elements $a_{1,N}, \ldots, a_{m,N}$ of $\mathscr{A}$ are given. Assume that for every nonempty subset $V \subset \{1, \ldots, m\}$ and for every k there exists limit*

$$\lim_{N\to\infty} N \cdot \phi_k\Big(\prod_{p\in V} a_{p,N}\Big) = q_k(V).$$

Set

$$S_{p,N} = (a_{p,N}, 1) + (a_{p,N}, 2) + \cdots + (a_{p,N}, N).$$

Then for every $k \geq 0$

$$\lim_{N\to\infty} \hat{\phi}_k(S_{1,N} S_{2,N} \cdots S_{m,N}) = \sum_{\pi\in\mathrm{NC}(m)} \prod_{V\in\pi} q_{k+d(V)}(V).$$

Proof. We follow ideas of Speicher [S1].

First of all note that in view of the formula preceding Lemma 1 we have

$$\lim_{N\to\infty} N \cdot R_k(V(a_{1,N}, \ldots, a_{m,N})) = q_k(V).$$

For a sequence $\mathbf{i} = (i_1, \ldots, i_m)$ we denote by $\pi_{\mathbf{i}}$ the partition of $\{1, \ldots, m\}$ given by $p\pi_{\mathbf{i}}q$ iff $i_p = i_q$. Consider

$$\hat{\phi}_k(S_{1,N} \cdots S_{m,N}) = \sum_{i_1,\ldots,i_m\in\{1,\ldots,N\}} \hat{\phi}_k((a_{1,N}, i_1) \cdots (a_{m,N}, i_m)).$$

Denoting by $\hat{\phi}_k(\mathbf{i}; N)$ the summand corresponding to $\mathbf{i} = (i_1, \dots, i_m) \in \{1, \dots, N\}^m$ we get from Theorem 1

$$\hat{\phi}_k(\mathbf{i}; N) = \sum_{\sigma \in \mathrm{NC}(\mathbf{i})} \prod_{V \in \sigma} R_{k+d(V)}(V(a_{1,N}, \dots, a_{m,N})),$$

which implies, for $\pi = \pi_{\mathbf{i}}$, $n = |\pi_{\mathbf{i}}|$,

$$\lim_{N \to \infty} N^n \cdot \hat{\phi}_k(\mathbf{i}; N) = \begin{cases} \prod_{V \in \pi} q_{k+d(V)}(V) & \text{if } \pi \in \mathrm{NC}(m) \\ 0 & \text{otherwise.} \end{cases}$$

Note that if $\pi_{\mathbf{i}} \in \mathrm{NC}(m)$ then $\pi_{\mathbf{i}} \in \mathrm{NC}(\mathbf{i})$ and if $\mathbf{i}, \mathbf{j} \in \{1, \dots, N\}^m$ and $\pi_{\mathbf{i}} = \pi_{\mathbf{j}}$ then $\hat{\phi}_k(\mathbf{i}; N) = \hat{\phi}_k(\mathbf{j}; N)$. Denoting this common value by $\hat{\phi}_k(\pi_{\mathbf{i}}; N)$ we note that for a fixed partition π of $\{1, \dots, m\}$ there are exactly $A(\pi, N) = N(N-1)\dots(N-|\pi|+1)$ sequences $\mathbf{i} \in \{1, \dots, N\}^m$ with $\pi_{\mathbf{i}} = \pi$. Hence

$$\hat{\phi}_k(S_{1,N} \cdots S_{m,N}) = \sum_{\pi} A(\pi; N) \hat{\phi}_k(\pi; N),$$

where the sum is taken over all partitions π of $\{1, \dots, m\}$, and consequently

$$\lim_{N \to \infty} \hat{\phi}_k(S_{1,N} \cdots S_{m,N}) = \sum_{\pi \in \mathrm{NC}(m)} \prod_{V \in \pi} q_{k+d(V)}(V).$$ □

For a nonnegative integer m we will denote by $\mathrm{NC}_2(m)$ the set of all partitions $\pi \in \mathrm{NC}(m)$ satisfying $|V| = 2$ for every block $V \in \pi$. Note that if m is odd then $\mathrm{NC}_2(m)$ is empty.

Corollary 3 (central limit theorem). *Let* $a_1, \dots, a_m \in \mathscr{A}$ *with* $\phi_k(a_p) = 0$ *for every* $k \geq 0$, $1 \leq p \leq m$ *and set*

$$S_{p,N} = \frac{1}{\sqrt{N}} \left[(a_p, 1) + \cdots + (a_p, N) \right].$$

Then

$$\lim_{N \to \infty} \hat{\phi}_k(S_{1,N} \cdots S_{m,N}) = \sum_{\pi \in \mathrm{NC}_2(m)} \prod_{\substack{V \in \pi \\ V = \{p,q\}, p<q}} \phi_{k+d(V)}(a_p a_q)$$

if m *is even and the limit equals* 0 *if* m *is odd.*

Proof. Putting $a_{p,N} = \frac{1}{\sqrt{N}} a_p$ we have $\phi_k(a_{p,N}) = 0$, $N \cdot \phi_k(a_{p,N} a_{q,N}) = \phi_k(a_p a_q)$, and for $V \subset \{1, \dots, m\}$, with $|V| \geq 3$, $\lim_{N \to \infty} N \cdot \phi_k (\prod_{p \in V} a_{p,N}) = 0$. □

Corollary 4 (Poisson limit theorem). *Assume that* $a_1, a_2, \ldots \in \mathscr{A}$ *with*

$$\lim_{N\to\infty} N \cdot \phi_k(\underbrace{a_N \cdots a_N}_{s \text{ times}}) = A_k$$

for every $k \geq 0$ *and* $1 \leq s \leq m$. *Then for*

$$S_N = (a_N, 1) + \cdots + (a_N, N)$$

we have

$$\lim_{N\to\infty} \hat{\phi}_k(\underbrace{S_N \cdots S_N}_{m \text{ times}}) = \sum_{\pi\in \mathrm{NC}(m)} \prod_{V\in\pi} A_{k+d(V)}.$$

Proof. For every $\emptyset \neq V \subset \{1, \ldots, m\}$ we have $q_k(V) = A_k$. □

8. Orthogonal polynomials

In this section we are going to describe the orthogonal polynomials related to the limit theorems. Let $\{P_n(x)\}_{n=0}^{\infty}$ be a sequence of polynomials over a field K, P_n of degree n, satisfying the following recurrence relation: $P_0 = 1$,

$$x P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x), \quad n \geq 0$$

(under convention that $P_{-1} \equiv 0$), where $\alpha_n, \beta_n, \gamma_n \in K$ and let Φ be a linear functional on $K[x]$ satisfying $\Phi(1) = 1$ and $\Phi(P_n) = 0$ for $n \geq 1$. In this situation we say that the sequence $\{P_n\}_{n=0}^{\infty}$ is *orthogonal* with respect to Φ. Note that the recurrence relation implies that $\Phi(x^m P_n(x)) = 0$, and, consequently, $\Phi(P_m(x)P_n(x)) = 0$, whenever $0 \leq m < n$. Set

$$\mathrm{NC}_{1,2}(m) = \{\pi \in \mathrm{NC}(m) : |V| \leq 2 \text{ for every } V \in \pi\}.$$

The following combinatorial formula (see [AB, Vi]) allows us to evaluate the moments $\Phi(x^m)$ of Φ.

Theorem 4. *For every* $m \geq 1$ *the following holds*

$$\Phi(x^m) = \sum_{\mathrm{NC}_{1,2}(m)} \prod_{\substack{V\in\pi \\ |V|=2}} \left(\alpha_{d(V,\pi)}\gamma_{d(V,\pi)}\right) \prod_{\substack{V\in\pi \\ |V|=1}} \beta_{d(V,\pi)}\,. \qquad \square$$

(*a*) *The central limit theorem* (*cf.* [*CD, CDI*]). For a sequence $\{A_k\}_{k=0}^{\infty}$, $A_k \in K$, we define a functional Φ_c on $K[x]$ by

$$\Phi_c(x^m) = \sum_{\pi \in \mathrm{NC}_2(m)} \prod_{V \in \pi} A_{d(V,\pi)}$$

We see immediately from Theorem 4 that the related sequence of monic orthogonal polynomials is given by: $P_0 \equiv 1$;

$$x P_n(x) = P_{n+1}(x) + A_{n-1} P_{n-1}(x) \quad \text{for } n \geq 0$$

($P_{-1} \equiv 0$). Note that the class of such sequences of polynomials, where $\{A_n\}_{n=0}^{\infty}$ ranges over all bounded sequences of positive numbers, corresponds to the class of all compactly supported symmetric probability measures on $\mathbf{R}$.

(*b*) *The Poisson limit theorem*. For this case we need to replace the sum over all noncrossing partitions $\mathrm{NC}(m)$ by a sum over $\mathrm{NC}_{1,2}(m)$.

Lemma 2. *For any sequence $\{A_k\}_{k=0}^{\infty}$ of numbers and for arbitrary integer $m \geq 1$ the following equality holds:*

$$\sum_{\sigma \in \mathrm{NC}(m)} \prod_{B \in \sigma} A_{d(B,\sigma)} = \sum_{\pi \in \mathrm{NC}_{1,2}(m)} \prod_{\substack{V \in \pi(o) \\ |V|=1}} A_0 \prod_{\substack{V \in \pi \\ |V|=2}} A_{d(V,\pi)} \prod_{\substack{V \in \pi(i) \\ |V|=1}} (A_{d(V,\pi)} + 1).$$

Proof. For a partition $\sigma \in \mathrm{NC}(m)$ we define a partition $\pi = \Lambda(\sigma)$ by replacing every block $V = \{k_1, k_2, \ldots, k_s\} \in \sigma$, $k_1 < k_2 < \cdots < k_s$, with $s \geq 2$, by one two-element block $\{k_1, k_s\}$ and $s - 2$ one-element blocks $\{k_2\}, \{k_3\}, \ldots, \{k_{s-1}\}$. Hence if $1 \leq p < q \leq m$ then

$$p \overset{\pi}{\sim} q \text{ iff there exists } V = \{k_1, k_2, \ldots, k_s\} \in \sigma \text{ with}$$
$$p = k_1 < k_2 < \cdots < k_s = q.$$

It is easy to see that

1) $\Lambda : \mathrm{NC}(m) \to \mathrm{NC}_{1,2}(m)$,

2) $\Lambda(\pi) = \pi$ for $\pi \in \mathrm{NC}_{1,2}(m)$.

Now let us fix $\pi \in \mathrm{NC}_{1,2}(m)$. For $V \in \pi$ with $|V| = 2$ set $S(V) = \{k : \{k\} \in \pi \text{ and } \{k\}'(\pi) = V\}$. If $\sigma \in \mathrm{NC}(m)$ and $\Lambda(\sigma) = \pi$ then the only difference between π and σ is that some of the one-element inner blocks $\{k\} \in \pi(i)$ can be in σ joined with their successors $\{k\}'(\pi)$. Therefore the class $\Lambda^{-1}(\pi) = \{\sigma \in \mathrm{NC}(m) : \Lambda(\sigma) = \pi\}$ can be described as follows.

Let $V_1, \dots, V_r$ be the two-element blocks of π. For a sequence $(S_1, \dots, S_r)$ of subsets $S_j \subset S(V_j)$ we define $\sigma = \pi(S_1, \dots, S_r)$ by

$$p \overset{\sigma}{\sim} q \text{ iff } p = q \text{ or } p, q \in V_j \cup (S(V_j) \setminus S_j) \quad \text{for some } 1 \le j \le r$$

(in particular $\pi(S(V_1), \dots, S(V_r)) = \pi$). Then

$$\Lambda^{-1}(\pi) = \{\pi(S_1, \dots, S_r) : S_1 \subset S(V_1), \dots, S_r \subset S(V_r)\}.$$

Therefore, putting $D = |\{k : \{k\} \in \pi(o)\}|$, $d_j = d(V_j, \pi)$,

$$\begin{aligned}
\sum_{\sigma \in \Lambda^{-1}(\pi)} \prod_{B \in \sigma} A_{d(B,\sigma)} &= A_0^D \sum_{\substack{S_1 \subset S(V_1) \\ \cdots \\ S_r \subset S(V_r)}} \prod_{j=1}^{r} \left(A_{d_j} A_{d_j+1}^{|S_j|} \right) \\
&= A_0^D \prod_{j=1}^{r} \left(A_{d_j} \sum_{S \subset S(V_j)} A_{d_j+1}^{|S|} \right) \\
&= A_0^D \prod_{j=1}^{r} \left(A_{d_j} (A_{d_j+1} + 1)^{|S(V_j)|} \right) \\
&= \prod_{\substack{V \in \pi(o) \\ |V|=1}} A_0 \cdot \prod_{\substack{V \in \pi \\ |V|=2}} A_{d(V,\pi)} \cdot \prod_{\substack{V \in \pi(i) \\ |V|=1}} (A_{d(V,\pi)} + 1),
\end{aligned}$$

which completes the proof. □

Now for a sequence $A_0, A_1, \ldots \in K$ we define $\Phi_P : K[x] \to K$ by

$$\Phi_P(x^m) = \sum_{\pi \in \mathrm{NC}(m)} \prod_{V \in \pi} A_{d(V,\pi)}.$$

Using Lemma 2 and Theorem 4 we see that the following sequence of monic polynomials is orthogonal for Φ_P: $P_0 \equiv 1$, $P_1(x) = x - A_0$,

$$x P_n(x) = P_{n+1}(x) + (A_n + 1) P_n(x) + A_{n-1} P_{n-1}(x), \quad \text{for } n \ge 1.$$

References

[AB] Accardi, L., Bożejko, M.: Interacting Fock spaces and Gaussianization of probability measures, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **1** no. 4 663–670 (1998)

[BS] Bożejko, M., Speicher, R.: ψ-independent and symmetrized white noises, in Quantum Probability and Related Topics, vol. VI, edited by L. Accardi, World Scientific 219–236 (1991)

[BLS] Bożejko, M., Leinert, M., Speicher, R.: Convolution and limit theorems for conditionally free random variables, Pacific J. Math. **175** no. 2 357–388 (1996)

[CD] Cabanal-Duvillard, T.: La σ-independance, 1993, preprint

[CDI] Cabanal-Duvillard, T., Ionescu, V.: Un Théorème central limite pour des variables aléatoires non-commutatives, C. R. Acad. Sci. Paris, **325** Série I, 1117–1120 (1997)

[K] Kreweras, G.: Sur les partitions noncroisées d'un cycle, Discrete Math. **1**, 333–350 (1972)

[M] Młotkowski, W.: Operator-valued version of conditionally free product, preprint.

[S1] Speicher, R.: A new example of "indepenence" and "white noise", Probab. Th. Rel. Fields, **84**, 141–159 (1990)

[S2] Speicher, R.: Multiplicative functions on the lattice of non-crossing partitions and free convolution, Math. Ann. **298**, 611–628 (1994)

[S3] Speicher, R.: Combinatorial theory of the free product with amalgamation and operator-valued free probability theory. Habilitationsschrift, Heidelberg, 1994.

[SW] Speicher, R., Woroudi, R.: Boolean Convolution, Fields Institute Communications **12**, 267–279 (1997)

[Vi] Viennot, G.: Une théorie combinatoire des polynômes orthogonaux généraux, Lecture Notes, UQAM, 1983.

[V] Voiculescu, D.: Symmetries of some reduced free product C^*–algebras, in Operator Algebras and their Connection with Topology and Ergodic Theory, Busteni, Romania, 1983, Lecture Notes in Mathematics **1132**, Springer Verlag, Heidelberg 1985.

[VDN] Voiculescu, D., Dykema, K. J., Nica, A.: Free Random Variables, CRM Monograph Series, Volume 1, 1992.