A SIMPLE CCC NON-SEPARABLE SPACE WITH SMALL MEASURES

PIOTR BORODULIN–NADZIEJA

Abstract. We show that in the model obtained by adding a Cohen real there is a ccc non-separable space which does not carry a non-separable measure.

In [Bel96] Bell proved that $\mathsf{MA}(\omega_1)$ implies existence of a compact ccc non-separable space which does not map continuously onto $[0,1]^{\omega_1}$. In fact in this construction only an assumption on existence of certain types of gaps was used.

In this note we show that Bell's space satisfies a stronger condition: it does not carry a non-separable measure (see below for the precise definition). We do it by translating his *total ideals space* technique into the language of Boolean algebras, like in [BND].

Although it can be shown that every space constructed by the above technique (from towers or (pre-)gaps) carries only separable measures, we prove it only in the model obtained by adding a Cohen real, since this assumption simplifies the proof. The proof for the general case can be easily found by adapting the (slightly technical) proof of [BND, Theorem 6.7].

Recall that a family $(L_\alpha, R_\alpha)_{\alpha<\kappa}$, where L_α, $R_\alpha \subseteq \omega$ for each α is a pre-gap if

- $L_\alpha \subseteq^* L_\beta$, $R_\alpha \subseteq^* R_\beta$;
- $L_\alpha \cap R_\alpha = \emptyset$

for each α, $\beta \in \kappa$. If additionally

- there is no $L \subseteq \omega$ such that $L_\alpha \subseteq^* L$ and $L \cap R_\alpha =^* \emptyset$ for each α,

then $(L_\alpha, R_\alpha)_{\alpha<\kappa}$ is *a gap*. If there is L as above, we say that L *interpolates* $(L_\alpha, R_\alpha)_{\alpha<\kappa}$. A gap satisfies *condition (K)* if

$$L_\alpha \cap R_\beta \neq \emptyset \neq L_\beta \cap R_\alpha$$

for each $\alpha, \beta < \kappa$. By a *sub-gap* of $(L_\alpha, R_\alpha)_{\alpha<\kappa}$ we mean $(L_\alpha, R_\alpha)_{\alpha\in\Lambda}$ for some (usually cofinal) $\Lambda \subseteq \kappa$. Confront [Yor03] for more information concerning gaps.

For a pre-gap $\mathcal{G} = (L_\alpha, R_\alpha)_{\alpha<\kappa}$ define a compact Hausdorff space in the following way. For a pair (L, R) of subsets of ω let

$$\rho(L,R) = \{x \in 2^\omega : x(n) = 0 \text{ for each } n \in L \text{ and } x(n) = 1 \text{ for each } n \in R\}.$$

Notice that $\rho(L,R)$ is a closed subset of the Cantor space. Now, let $\mathfrak{A}$ be the Boolean algebra generated by all sets of the form $\rho(L,R)$, where $L =^* L_\alpha$ and $R =^* R_\alpha$ for some $\alpha < \kappa$. Let K be the Stone space of $\mathfrak{A}$.

Remark 1. *The space K is Hausdorff, zero-dimensional and compact (as a Boolean space).*

The following two propositions are contained (in a slightly different language) in [Bel96].

Proposition 2. *The family of sets of the form $\rho(L,R)$ for $L =^* L_\alpha$ and $R =^* R_\alpha$ for some $\alpha < \kappa$ is a π-base of K.*

Proof. Sets of the form

$$U = \rho(A_0, B_0) \cap \cdots \cap \rho(A_n, B_n) \cap \rho(A'_0, B'_0)^c \cap \cdots \cap \rho(A'_m, B'_m)^c$$

form a base of K. If a set as above is non-void, then for each $i \leq m$ we can find

$$b_i \in A'_i \setminus (A_0 \cup \cdots \cup A_n)$$

and

$$a_i \in B'_i \setminus (B_0 \cup \cdots \cup B_n).$$

Let

$$L = A_0 \cup \cdots \cup A_n \cup \{a_0, \ldots, a_m\}$$

and

$$R = B_0 \cup \cdots \cup B_n \cup \{b_0, \ldots, b_m\}.$$

It is straightforward to check that $\rho(L, R) \subseteq U$ and $L =^* L_\alpha$, $R =^* R_\alpha$ for some $\alpha < \kappa$. □

Recall that a zero-dimensional space has *the κ-Knaster property* if each collection of κ clopen subsets has a centered subfamily of cardinality κ.

Lemma 3. *The space K defined as above is*

(1) separable if and only if $\mathcal{G}$ is not a gap,
(2) ccc if and only if $\mathcal{G}$ does not contain an uncountable sub-gap satisfying the condition (K).

In fact, if $\mathcal{G}$ is a gap of size κ, then K does not have a κ-Knaster property.

Proof. To show (1) suppose first that there is a set $I \subseteq \omega$ interpolating $\mathcal{G}$, i.e. such that $L_\alpha \subseteq^* I$ and $R_\alpha \cap I =^* \emptyset$ for each $\alpha < \kappa$. Consider the family $\{I_n \colon n \in \omega\}$ of all sets almost equal to I. For each n let $x_n \in K$ be such that $\{c \in 2^\omega \colon c(k) = 0\}$ belongs to x_n if and only if $k \in I_n$. Let $L =^* L_\alpha$, $R =^* R_\alpha$ for certain $\alpha < \kappa$. Consider n such that $L \subseteq I_n$. Then $x_n \in \rho(L, R)$. Hence, Remark 2 implies that K is separable.

On the other hand, we will show that if $\mathcal{G}$ is a gap, then there is no $x \in K$ which belongs to κ many elements of the form $\rho(L, R)$ for $L =^* L_\alpha$, $R =^* R_\alpha$. This would imply that K is not separable (and, in fact, that $\mathfrak{A}$ does not posses the κ–Knaster property). Assume that there is such $x \in K$ and consider the set

$$I = \{n \in \omega \colon \{c \in 2^\omega \colon c(n) = 0\} \in x\}.$$

Let $\alpha < \kappa$. There is $\alpha' > \alpha$ such that $x \in \rho(L, R)$ for $L =^* L_{\alpha'}$ and $R =^* R_{\alpha'}$. But this means that $L \subseteq I$ and $I \cap R = \emptyset$. So, $L_\alpha \subseteq^* I$ and $R_\alpha \cap I =^* \emptyset$. Since α was chosen arbitrarily, I interpolates $\mathcal{G}$.

To prove (2) assume that there is an uncountable $\Lambda \subseteq \kappa$ such that $(L_\alpha \cup L_\beta) \cap (R_\alpha \cup R_\beta) \neq \emptyset$ for each $\alpha < \beta \in \Lambda$. Plainly, $\{\rho(L_\alpha, R_\alpha) \colon \alpha \in \Lambda\}$ is pairwise disjoint and thus K is not ccc.

To prove the converse suppose K is not ccc. By Remark 2 we may assume that there is an uncountable $\Lambda \subseteq \kappa$ such that

$$\{\rho(L, R) \colon L =^* L_\alpha,\ R =^* R_\alpha \text{ for some } \alpha \in \Lambda\}$$

is pairwise disjoint. We may assume without loss of generality that the above family consists of sets of the form $\rho(L_\alpha \bigtriangleup F, R_\alpha \bigtriangleup G)$ for $\alpha \in \Lambda$ for fixed disjoint finite sets F and G. But we have $\alpha < \beta$ in Λ such that $(L_\alpha \cup L_\beta) \cap (R_\alpha \cup R_\beta) = \emptyset$. This means that $\rho(L_\alpha, R_\alpha) \cap \rho(L_\beta, R_\beta) \neq \emptyset$ and so $\rho(L_\alpha \bigtriangleup F, R_\alpha \bigtriangleup G) \neq \emptyset$. □

Note that for each space constructed in the above way there is no continuous surjection $f\colon K \to [0,1]^{\omega_1}$. Indeed, one can show that each filter on $\mathfrak{A}$ can be extended to an ultrafilter by countably many sets. This means that every closed subset of K has a point which is a relative G_δ. By Shapirovsky theorem (see [Šap80]) this implies that K cannot be continuously mapped onto $[0,1]^{\omega_1}$. In fact, one can shown that each space like above carries only separable measures (just mimic the proof of [BND, Theorem 6.7]). Recall that a measure μ on K is separable if $l_1(K)$ is separable. In the zero-dimensional case it is equivalent to saying that the (pseudo-)metric $d_\mu(A,B) = \mu(A \triangle B)$ defined on the Boolean algebra of clopen subsets of K is separable. Note that if K can be mapped continuously onto $[0,1]^{\omega_1}$, then it carries a non-separable measure.

Below we show that in a model obtained by adding a Cohen real there is a gap $\mathcal{G}$ which does not contain an uncountable sub-gap satisfying the condition (K) and for which it is particularly easy to show that each measure defined on the appropriate space is separable.

Proposition 4. *If $(L_\alpha, R_\alpha)_{\alpha<\omega_1}$ is a gap, and c is a Cohen real in the extension (seen as a subset of ω). Then for each uncountable $X \subseteq \omega_1$ in the extension there are $\alpha < \beta \in X$ such that*

$$L_\alpha \cap c \subseteq L_\beta \cap c \text{ and } R_\alpha \cap c \subseteq R_\beta \cap c.$$

Proof. Since the Cohen forcing notion is countable we can assume without loss of generality that X belongs to the ground model. Let $p \in 2^n$ and let $\alpha < \beta$ be such that $L_\alpha \cap n = L_\beta \cap n$ and $R_\alpha \cap n = R_\beta \cap n$. Let m be such that $L_\alpha \setminus m \subseteq L_\beta$ and $R_\alpha \setminus m \subseteq R_\beta$. Then $q \in 2^m$ such that $q|n = p|n$ and $q(i) = 0$ for each $i > n$ forces that $L_\alpha \cap c \subseteq L_\beta \cap c$ and $R_\alpha \cap c \subseteq R_\beta \cap c$. □

Clearly, Proposition 4 implies that the gap $\mathcal{G} = (L_\alpha \cap c, R_\alpha \cap c)_\alpha$ does not have a cofinal sub-gap with property (K) in the Cohen extension (to see that it is a gap, notice that c has an infinite intersection with every infinite subset of ω from the ground model). So, if K is the space constructed in the way described above from $\mathcal{G}$, then K is non-separable and ccc. We will show the following:

Proposition 5. *If K is as above and μ is defined on K, then μ is separable.*

Proof. We work in the Cohen extension. Assume we have a non-separable μ defined on K. Then there is an uncountable set $\mathcal{A} \subseteq \mathfrak{A}$ and $\varepsilon > 0$ such that

$$\mu(A \triangle B) > \varepsilon$$

for each A, $B \in \mathcal{A}$. Without loss of generality we can assume that $\mathcal{A}$ consists of the generators of $\mathfrak{A}$. So, there is an uncountable $X_0 \subseteq \omega_1$ and collections $(L'_\alpha)_{\alpha\in X_0}$ and $(R'_\alpha)_{\alpha\in X_0}$ such that $L'_\alpha =^* L_\alpha$, $R'_\alpha =^* R_\alpha$ for each $\alpha \in X_0$ and

$$\mu(\rho(L'_\alpha \cap c, R'_\alpha \cap c) \triangle \rho(L'_\beta \cap c, R'_\beta \cap c)) > \varepsilon$$

for each α, $\beta \in X_0$.

Define $X \subseteq X_0$ to be an uncountable set such that for each α, $\beta \in X$

(1) $L'_\alpha \triangle L_\alpha = L'_\beta \triangle L_\beta$;
(2) $R'_\alpha \triangle R_\alpha = R'_\beta \triangle R_\beta$;
(3) $|\mu(\rho(L'_\alpha, R'_\alpha)) - \mu(\rho(L'_\beta, R'_\beta))| < \varepsilon/2$.

By Proposition 4 there is $\alpha < \beta \in X$ such that

$$L_\alpha \cap c \subsetneq L_\beta \cap c \text{ and } R_\alpha \cap c \subsetneq R_\beta \cap c.$$

Properties (1) and (2) imply that then

$$L'_\alpha \cap c \subsetneq L'_\beta \cap c \text{ and } R'_\alpha \cap c \subsetneq R'_\beta \cap c.$$

Hence, $\rho(L'_\beta \cap c, R'_\beta \cap c) \subsetneq \rho(L'_\alpha \cap c, R'_\alpha \cap c)$. But this means that

$$\mu(\rho(L'_\alpha \cap c, R'_\alpha \cap c)) - \mu(\rho(L'_\beta \cap c, R'_\beta \cap c)) > \varepsilon,$$

a contradiction with the property (3). □

So, finally we obtain the following theorem:

Theorem 6. *In a model obtained by adding a single Cohen real there is a compact Hausdorff space K such that*

- *K is not separable (it does not even have ω_1-Knaster property),*
- *K is ccc,*
- *K does not carry a non-separable measure.*

References

[Bel96] Murray Bell. A compact ccc non-separable space from a Hausdorff gap and Martin's axiom. *Comment. Math. Univ. Carolin.*, 37(3):589–594, 1996.

[BND] Piotr Borodulin-Nadzieja and Mirna Džamonja. On isomorphism problem for measures on Boolean algebras. *preprint*.

[Šap80] B. È. Šapirovskiĭ. Mappings on Tihonov cubes. *Uspekhi Mat. Nauk*, 35(3(213)):122–130, 1980. International Topology Conference (Moscow State Univ., Moscow, 1979).

[Yor03] Teruyuki Yorioka. Distinguishing types of gaps in $\mathbb{P}(\omega)$/fin. *J. Symbolic Logic*, 68(4):1261–1276, 2003.

Instytut Matematyczny, Uniwersytet Wrocławski
E-mail address: pborod@math.uni.wroc.pl