# Ideals with bases of unbounded Borel complexity

Piotr Borodulin–Nadzieja*, Szymon Głąb

**Abstract**

We present several naturally defined $\sigma$–ideals which have Borel bases but, unlike for the classical examples, these bases are not of bounded Borel complexity. We investigate set-theoretic properties of such $\sigma$–ideals.

## 1 Introduction

Consider the $\sigma$–ideal of subsets of the plane consisting of sets which for all $\varepsilon > 0$ can be covered by an open set whose all vertical sections have measure less than $\varepsilon$. It seems natural to suppose that this ideal has similar properties to those of the ideal of null sets, e.g. Gabriel Mokobodzki conjectured that under Martin's Axiom the additivity of this ideal equals continuum (see [9], Problem 32Rd). It turned out to be not true. Cichoń and Pawlikowski proved in [6] that the additivity of this ideal, called there Mokobodzki ideal, equals $\omega_1$ in `ZFC`.

In the same paper Cichoń and Pawlikowski considered also $\sigma$–ideals of subsets of $[0,1]^2$ which can be covered by Borel sets whose every vertical section is small (i.e. of Lebesgue measure 0 or meager). These ideals also have additivity $\omega_1$. Cichoń and Pawlikowski observed one more interesting property: these ideals do not have Borel bases of bounded Borel complexity (although, clearly, they have Borel bases). In other words, $\mathcal{M}_\alpha \subsetneq \mathcal{M}$ for each $\alpha$, where $\mathcal{M}$ is such an ideal and $\mathcal{M}_\alpha$ is the $\sigma$–ideal generated by $\mathcal{M} \cap \Sigma^0_\alpha$.

In this paper we consider certain modifications of Mokobodzki ideals, e.g. the $\sigma$–ideal of subsets of the plane which are small not only on vertical sections but also on horizontal ones or which are small *in every direction*. In Section 3 we prove that such $\sigma$-ideals also do not have Borel bases of bounded Borel complexity.

A set $A$ is in Mokobodzki ideal if its *every* section is small. One can ask if it is possible to change *every* to *almost every* and still having a $\sigma$-ideal with bases of unbounded Borel complexity. It can be done if we interpret the word "almost" correctly. Notice that ideals of sets whose almost every (with respect to Lebesgue measure or with respect to Baire category) section is small (Lebesgue null or meager, respectively) are the ideals of null or meager subsets of the real plane, and therefore they have bases of bounded Borel complexity (e.g. consisting of $\Pi^0_2$ or $\Sigma^0_2$ sets, respectively). However, in Section

*The first author was partially supported by the Polish Ministry of Science and Higher Education under grant no. N N201 418939.

2010 *Mathematics Subject Classification*: Primary 54A35, 28A05.

Keywords: ideals on the plane, Mokobodzki ideal, Fubini products, uniformization, property (M).

4 we prove that *every* can be changed to *almost every with respect to a $\sigma$-ideal having property (M).*

In Section 5 we investigate set-theoretic properties of the $\sigma$–ideals considered in the paper. In the last section we present a remark concerning ideals with property (M) and we state some open questions. Facts proved in the two last sections indicate that Mokobodzki ideals and other ideals considered here differ from $Null$ and $Meager$ in many aspects.

**Acknowledgements.** The authors would like to thank Jacek Cichoń for introducing them to the notion of Mokobodzki ideal, Piotr Zakrzewski for valuable suggestions, and an anonymous referee for the careful examination of the paper and for helpful comments.

# 2 Preliminaries

All terminology which is not explained here, can be found e.g. in [11] and [5]. By $\Sigma^0_\alpha(X)$, $\Pi^0_\alpha(X)$, Borel$(X)$, $\Sigma^1_1(X)$ and $\Pi^1_1(X)$ we mean the families of, respectively, $\Sigma^0_\alpha$, $\Pi^0_\alpha$, Borel, analytic and coanalytic subsets of a Polish space $X$. Usually, $X$ will be known from the context and we omit it. By $\pi_1\colon X\times Y\to X$ and $\pi_2\colon X\times Y\to Y$ we denote the projections on the first and on the second coordinates, i.e. $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for every $(x,y)\in X\times Y$. For $A\subseteq X\times Y$ define a vertical section of $A$ at a point $x$ as $A_x=\{y\in Y\colon (x,y)\in A\}$ and a horizontal section of $A$ at a point $y$ as $A^y=\{x\in X\colon (x,y)\in A\}$. In the paper we use several times the fact that if $A$ and $f$ are Borel and $f|A$ is injective, then $f(A)$ is Borel (see [11, 15.1]).

Let $\mathcal{I}$ be a $\sigma$–ideal on $Y$ and $\mathcal{J}$ be a $\sigma$–ideal on $X$. The Fubini product $\mathcal{J}\otimes\mathcal{I}$ is the $\sigma$–ideal on $X\times Y$ generated by the family $\{B\in \text{Borel}(X\times Y)\colon \{x\colon B_x\notin\mathcal{I}\}\in\mathcal{J}\}$. We say that a $\sigma$–ideal $\mathcal{I}$ is $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$ if $\{x\colon B_x\in\mathcal{I}\}\in\Pi^0_\alpha$ for every $B\in\Sigma^0_\alpha(X\times X)$. Let $A$ be a $\Sigma^0_\alpha$ subset of $[0,1]^2$. By [11, 22.22] the set $\{x\colon A_x \text{ is non-meager}\}$ is $\Sigma^0_\alpha$. Then $\{x\colon A_x \text{ is meager}\}=\{x\colon A_x \text{ is non-meager}\}^c$ is $\Pi^0_\alpha$. Using [11, 22.25] we obtain that $\{x\colon A_x \text{ is not null}\}$ is $\Sigma^0_\alpha$. Hence $\{x\colon A_x \text{ is null}\}$ is $\Pi^0_\alpha$. This shows that the ideals $\mathsf{Meager}$ and $\mathsf{Null}$ are $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$ for every $\alpha<\omega_1$. In a similar way we define properties Borel–on–Borel, $\Pi^1_1$–on–$\Sigma^1_1$ etc. Note that the ideals $\mathsf{Meager}$ and $\mathsf{Null}$ are $\Pi^1_1$–on–$\Sigma^1_1$ (see [11, 29.22 and 29.26]).

Let $\mathcal{J}$ be a $\sigma$-ideal. A family $\mathcal{B}\subseteq\mathcal{J}$ is called a base of $\mathcal{J}$ if any set of $\mathcal{J}$ is contained in some set from $\mathcal{B}$. If there is a base of $\mathcal{J}$ consisting of Borel or $\Sigma^0_\alpha$ sets, then we say that $\mathcal{J}$ has a Borel or $\Sigma^0_\alpha$ base, respectively. For an ordinal number $\alpha\le\omega_1$, by $\mathcal{J}_\alpha$ we denote the $\sigma$–ideal generated by $\mathcal{J}\cap\Sigma^0_\alpha$. Note that if $\mathcal{J}$ has a Borel base, then $\mathcal{J}=\bigcup_{\alpha<\omega_1}\mathcal{J}_\alpha$.

Let $\mathcal{I}$ be a $\sigma$–ideal of subsets of an uncountable Polish space $X$. We say that $A\subseteq X^2$ is in the $\sigma$–ideal $\mathcal{M}(\mathcal{I})$ if there is a Borel set $B\supseteq A$ such that $B_x\in\mathcal{I}$ for every $x\in X$. We write $\mathcal{M}_\alpha(\mathcal{I})$ instead of $(\mathcal{M}(\mathcal{I}))_\alpha$. We will say that a $\sigma$–ideal $\mathcal{J}$ with a Borel base has *the complex Borel base property* if for every $\alpha<\omega_1$ we have $\mathcal{J}_\alpha\subsetneq\mathcal{J}$. In [6] the authors mentioned that $\mathcal{M}(\mathsf{Null})$ and $\mathcal{M}(\mathsf{Meager})$ have the complex Borel base property.

For a $\sigma$–ideal $\mathcal{I}$ of subsets of $X^2$ and a family of functions $\mathcal{F}\subseteq X^X$ define a $\sigma$–ideal $\mathcal{M}(\mathcal{F},\mathcal{I})$ on $X^2$ in the following way: $Y\subseteq X^2$ belongs to $\mathcal{M}(\mathcal{F},\mathcal{I})$ whenever $Y\in\mathcal{M}(\mathcal{I})$ and $Y$ can be covered by a Borel set $B\subseteq X^2$ such that $\{x\colon (x,f(x))\in B\}\in\mathcal{I}$ for every $f\in\mathcal{F}$. As before, $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I})=(\mathcal{M}(\mathcal{F},\mathcal{I}))_\alpha$. Note that if $\mathcal{F}=\emptyset$, then $\mathcal{M}(\mathcal{F},\mathcal{I})=\mathcal{M}(\mathcal{I})$; if $\mathcal{F}$ consists of all constant functions $f\equiv y$, then $\{x\colon (x,f(x))\in$

$B\} = \{x\colon (x,y) \in B\} = B^y$ which means that all horizontal sections $B^y$ of $B$ are in $\mathcal{I}$; if $\mathcal{F}$ is a family of all linear mappings, then $\mathcal{M}(\mathcal{F},\mathcal{I})$ consists of all subsets of real plane which can be covered by Borel set $\mathcal{I}$-small in every direction, etc. Hence, $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I})$ seems to be a natural generalization of $\mathcal{M}_\alpha(\mathcal{I})$. In Section 3 we discuss the properties of such $\sigma$–ideals and we show that, under natural assumptions on $\mathcal{F}$, they have the complex Borel base property.

One of the well–known uniformization theorems states that if every nonempty section of a Borel set $A \subseteq X^2$ is not in $\mathcal{I}$ (where $X$ is an uncountable Polish space and $\mathcal{I}$ is a Borel–on–Borel ideal), then $A$ has a Borel uniformization, i.e. a Borel function $f\colon \pi_1[A] \to X$ such that $f(x) \in A_x$ for every $x \in \pi_1[A]$ [11, 18.6]. Recently, Petr Holický proved a theorem which gives an information about the Borel class of this uniformization. Here we state a simplified version of Holický's result needed for our purposes. It immediately follows from [10, Theorem 3.3] and [10, Theorem 3.4]. Recall that a function $f\colon X \to Y$ is $\Sigma^0_\alpha$–measurable if for every $E \in \Sigma^0_1(Y)$, the set $f^{-1}[E] \in \Sigma^0_\alpha(X)$. The graph of $\Sigma^0_\alpha$–measurable function belongs to $\Pi^0_\alpha(X \times Y)$ if $X$ and $Y$ are Polish (see, e.g. [12] §31, VII, Thm 1).

**Theorem 2.1 (Holický)** *Suppose $X$ is an uncountable Polish space. Let $\mathcal{I}$ be a $\sigma$–ideal of subsets of $X$ which is $\Sigma^0_\alpha$-on-$\Pi^0_\alpha$ for some $2 \le \alpha < \omega_1$ and which contains all singletons. Let $A \subseteq X^2$ be such that $A_x \notin \mathcal{I}$ for every $x \in \pi_1[A]$. If $A$ is of class $\Sigma^0_\alpha$, then there is a Borel function $F\colon \{0,1\}^\omega \times X \to X$ such that:*

$$\forall x \in \{0,1\}^\omega \ \ (y \mapsto F(x,y) \text{ is a } \Sigma^0_\alpha\text{-measurable uniformization of } A)$$

$$\forall y \in X \ \ (x \mapsto F(x,y) \text{ is continuous and 1-1}).$$

*In particular, there is a $\Sigma^0_\alpha$–measurable uniformization of $A$.*

The following theorem was proved in [6]. We repeat here its proof since we slightly modify its conclusion and we will use a similar argument later.

**Theorem 2.2 (Cichoń, Pawlikowski, Lemma 2.3 in [6])** *Assume $\mathcal{I}$ is a $\sigma$–ideal of subsets of an uncountable Polish space $X$ such that $X \notin \mathcal{I}$. For every $\alpha < \omega_1$ there is a $\Pi^0_\alpha$ set $A \subseteq X^2$ such that for every $M \in \mathcal{M}_\alpha(\mathcal{I})$ there is $x \in X$ such that $\emptyset \neq A_x \subseteq M^c_x$. If, additionally, $\mathcal{I}$ is $\Sigma^0_\alpha$-on-$\Pi^0_\alpha$, then we can assume that $A^c_x \in \mathcal{I}$ for every $x \in \pi_1[A]$.*

**Proof.** Let $\alpha < \omega_1$. By [11, 22.3] there is a universal set $U \subseteq X \times X^2$ for the pointclass $\Sigma^0_\alpha(X^2)$, i.e. for every $E \subseteq X^2$ such that $E \in \Sigma^0_\alpha$ there is $x \in X$ with $E = U_x$. Put $A = \{(x,y) \in X^2\colon (x,x,y) \notin U\}$. Clearly, $A$ is a $\Pi^0_\alpha$ subset of $X^2$. Let $M \in \mathcal{M}_\alpha(\mathcal{I})$. There is $E \in \Sigma^0_\alpha \cap \mathcal{M}_\alpha(\mathcal{I})$ such that $M \subseteq E$. Since $U$ is universal for $\Sigma^0_\alpha(X^2)$, there is $x_0 \in X$ with $E = U_{x_0}$. Since $A_{x_0} = X \setminus U_{(x_0,x_0)} = X \setminus (U_{x_0})_{x_0} = E^c_{x_0}$, then $A^c_{x_0} \in \mathcal{I}$.
Now assume that $\mathcal{I}$ is $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$. Then the set $\{x\colon A^c_x \in \mathcal{I}\}$ is $\Pi^0_\alpha$. Define

$$A' = A \cap (\{x\colon A^c_x \in \mathcal{I}\} \times X).$$

Of course $A'$ is $\Pi^0_\alpha$. Fix $x \in X$ such that $U_x \in \mathcal{M}_\alpha(\mathcal{I})$ and notice that $A_x = \{y\colon (x,x,y) \notin U\}$ and $U_{(x,x)} \in \mathcal{I}$. Therefore $A^c_x \in \mathcal{I}$ and $A_x = A'_x$. Consequently, for any $M \in \mathcal{M}_\alpha(\mathcal{I})$ there is $x \in X$ such that $A'_x \subseteq M^c_x$. Thus, we can assume without loss of generality that $A^c_x \in \mathcal{I}$ for every $x \in \pi_1[A]$.

■

# 3 Ideals $\mathcal{M}(\mathcal{F},\mathcal{I})$

In this section $(X,\cdot)$ stands for an uncountable Polish group. One can e.g. think about the group $X=\mathbb{R}$ with the addition or $X=2^\omega$ with the standard additive operation. Let $\mathcal{I}$ be either the $\sigma$–ideal of meager subsets of $X$ or a $\sigma$–ideal of null subsets of $X$ with respect to a right–invariant $\sigma$–finite measure on $X$.

We do not consider $\mathcal{I}$ in more general setting, since we will need many particular properties of the above ideals: Fubini property, ccc, right–invariance, $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$ for each $\alpha<\omega_1$. Zakrzewski's result from [15] implies that in the case $X=2^\omega$ the ideals $\mathsf{Null}$ and $\mathsf{Meager}$ are the only $\sigma$–ideals satisfying the above properties. This and other well–known results (see [8]) suggest that in case of arbitrary Polish group, if $\mathcal{I}$ satisfy the above properties then it is at least isomorphic to one of the above ideals.

If $\mathcal{I}$ is an ideal of null sets, then we will additionally assume that the measure is $\sigma$–finite, since we will need the following property: every Borel function $f\colon B\to X$, where $B\subseteq X$ is $\mathcal{I}$–positive and Borel, is continuous on an $\mathcal{I}$–positive $\Sigma^0_3$ subset of $B$. This property is satisfied under the above condition (see [11], Thm. 17.12) and in the case $\mathcal{I}$ is the ideal of meager subsets.

For the functions $f,\ g\colon X\to X$ let $(f\cdot g)(x)=f(x)\cdot g(x)$.

A family of functions $\mathcal{F}\subseteq X^X$ is *ubiquitous with respect to an ideal* $\mathcal{I}$ (or $\mathcal{I}$–ubiquitous) if for every Borel function $g\colon X\to X$ there is a Borel set $B\notin\mathcal{I}$ and a function $f\in\mathcal{F}$ such that $f|B=g|B$. The family of continuous functions is a natural example of $\mathsf{Null}$– and $\mathsf{Meager}$–ubiquitous family (it follows from Luzin Theorem [11, 17.12] and Nikodym Theorem [11, 8.38]). On the other hand, there are families of Borel functions $f\colon\mathbb{R}\to\mathbb{R}$ which are closed under the addition but are not ubiquitous neither with respect to $\mathsf{Null}$ nor to $\mathsf{Meager}$ ideals: the empty family, the constant functions, the linear functions. Note that also the family of polynomials is not ubiquitous neither with respect to $\mathsf{Null}$ nor to $\mathsf{Meager}$: e.g. the exponential function cannot equal to a polynomial on a set of with an accumulation point (as zeros of a holomorphic function must be isolated).

We will show that if a family $\mathcal{F}$ of Borel functions is left shift invariant and is not $\mathcal{I}$–ubiquitous, then any $\Pi^0_\alpha$ set with large sections and a big projection on $x$–axis have a uniformization with a graph of class $\Sigma^0_{\alpha+2}$ witnessing that $\mathcal{F}$ is not $\mathcal{I}$–ubiquitous.

**Lemma 3.1** *Assume that $A$ is a Borel subset of $X^2$ such that $A_x\notin\mathcal{I}$ for every $x\in\pi_1[A]\notin\mathcal{I}$. For each Borel mapping $h\colon X\to X$ we can find a Borel set $B\subseteq\pi_1[A]$ with $B\notin\mathcal{I}$ and $y\in X$ such that $y\cdot h(x)\in A_x$ for every $x\in B$.*

**Proof.** Define $\varphi\colon X^2\to X^2$ by $\varphi(x,y)=(x,y\cdot(h(x))^{-1})$. Then for every $x$

$$(\varphi[A])_x=(\{(x,y\cdot(h(x))^{-1})\colon (x,y)\in A\})_x=\{y\cdot(h(x))^{-1}\colon y\in A_x\}=A_x\cdot(h(x))^{-1}.$$

Since $\mathcal{I}$ is right invariant, then

$$(\varphi[A])_x\in\mathcal{I} \text{ iff } A_x\in\mathcal{I}$$

and, thus, $(\varphi[A])_x\notin\mathcal{I}$ for every $x\in\pi_1[A]\notin\mathcal{I}$. The set $\varphi[A]$ is Borel because $\varphi$ is a Borel one–to–one mapping. By the Fubini property of $\mathcal{I}$ there is $y\in X$ such that

$$B=(\varphi[A])^y\in \mathrm{Borel}\setminus\mathcal{I}.$$

Let $x\in B$. Then $(x,y)\in\varphi[A]$ and, therefore, $(x,y\cdot h(x)\cdot(h(x))^{-1})\in\varphi[A]$ but this means that $(x,y\cdot h(x))\in A$. So, $y\cdot h(x)\in A_x$ for every $x\in B$.

■

**Lemma 3.2** *Let $3 \le \alpha < \omega_1$. Assume $A \subseteq X^2$ is a $\Pi^0_\alpha$ set such that $A_x \notin \mathcal{I}$ for every $x \in \pi_1[A] \notin \mathcal{I}$. Then, for a Borel function $h\colon X \to X$ there is a countable collection $\mathcal{B}$ of pairwise disjoint Borel sets and a uniformization $f\colon \pi_1[A] \to X$ of $A$ such that*

*(i) $\pi_1[A] \setminus \bigcup \mathcal{B} \in \mathcal{I}$,*

*(ii) for every $B \in \mathcal{B}$ there is $y_B \in X$ such that $f|B = y_B \cdot h$,*

*(iii) $f$ is $\Sigma^0_{\alpha+1}$–measurable (and, consequently, its graph is $\Pi^0_{\alpha+1}$).*

**Proof.** Fix $A \subseteq X^2$ and $h\colon X \to X$ as above. Since $A \in \Sigma^0_{\alpha+1}$, we can use Theorem 2.1 to fix a $\Sigma^0_{\alpha+1}$–measurable uniformization $g\colon \pi_1[A] \to X$ of $A$. Use Lemma 3.1 to find a Borel set $B_0$ and $y_0$ such that $B_0 \subseteq \pi_1[A]$, $B_0 \notin \mathcal{I}$ and $y_0 \cdot h(x) \in A_x$ for each $x \in B_0$. We can assume that $(y_0 \cdot h)|B_0$ is continuous and $B_0$ is $\Sigma^0_3$ (more precisely, $\Pi^0_2$ if $\mathcal{I} = \mathsf{Meager}$, and $\Sigma^0_2$ if $\mathcal{I} = \mathsf{Null}$), shrinking $B_0$ if needed.

Assume now that we have constructed a family of pairwise disjoint $\Sigma^0_3$ sets $\{B_\xi \colon \xi < \beta\}$ and a family of points $\{y_\xi \colon \xi < \beta\}$ such that $B_\xi \subseteq \pi_1[A]$, $B_\xi \notin \mathcal{I}$, $y_\xi \cdot h(x) \in A_x$ for each $x \in B_\xi$, $(y_\xi \cdot h)|B_\xi$ is continuous and $B_\xi$ is $\Sigma^0_3$ .

If $Y = \pi_1[A] \setminus \bigcup_{\xi<\beta} B_\xi \notin \mathcal{I}$, then use Lemma 3.1 to find a Borel set $B_\beta$ and a point $y_\beta$ such that $B_\beta \subseteq \pi_1[A \cap (Y \times X)] = Y \cap \pi_1[A]$, $B_\beta \notin \mathcal{I}$, $y_\beta \cdot h(x) \in A_x$ for each $x \in B_\beta$, $(y_\beta \cdot h)|B_\beta$ is continuous and $B_\beta$ is $\Sigma^0_3$. Since $\mathcal{I}$ is ccc, there is $\beta < \omega_1$ such that $\pi_1[A] \setminus \bigcup_{\xi<\beta} B_\xi \in \mathcal{I}$. Let $\mathcal{B} = \{B_\xi \colon \xi < \beta\}$. Define $f\colon \pi_1[A] \to X$ in the following way. Let $f(x) = y_\xi \cdot h(x)$ if $x \in B_\xi$ and let $f(x) = g(x)$ for $x \in \pi_1[A] \setminus \bigcup \mathcal{B}$.

We have to verify that $f$ defined in this way is $\Sigma^0_{\alpha+1}$–measurable. Indeed, for every $\xi < \beta$ the function $h_\xi\colon B_\xi \to X$ defined by $h_\xi(x) = y_\xi \cdot h(x)$ is continuous on $B_\xi$. So, for $E \in \Sigma^0_1$ we have

$$f^{-1}[E] = \bigcup_{\xi<\beta} (h_\xi^{-1}[E] \cap B_\xi) \cup (g^{-1}[E] \setminus \bigcup \mathcal{B})$$

which is a countable union of $\Sigma^0_{\alpha+1}$ sets, so a $\Sigma^0_{\alpha+1}$ set.

■

Now we will use the above lemmas to prove that $\mathcal{M}(\mathcal{F}, \mathcal{I})$ has the complex base property under certain assumptions on $\mathcal{F}$.

**Theorem 3.3** *Let $\mathcal{F} \subseteq X^X$ be a family of Borel functions which is not $\mathcal{I}$-ubiquitous. Assume that $\mathcal{F}$ is left shift invariant, i.e. for any $f \in \mathcal{F}$ and $y \in X$ the function $x \mapsto y \cdot f(x)$ belongs to $\mathcal{F}$. Then $\mathcal{M}_{\alpha+2}(\mathcal{F}, \mathcal{I}) \setminus \mathcal{M}_\alpha(\mathcal{I}) \neq \emptyset$ for every $3 \le \alpha < \omega_1$.*

**Proof.** Let $3 \le \alpha < \omega_1$. Let $A$ be a $\Pi^0_\alpha$ set whose existence is guaranteed by Theorem 2.2.

Let $h$ be a Borel function witnessing that $\mathcal{F}$ is not $\mathcal{I}$-ubiquitous, i.e. there is no an $\mathcal{I}$–positive Borel set on which $h$ equals to a function from $\mathcal{F}$.

Use Lemma 3.2 for $A$ and $h$ to find a $\Sigma^0_{\alpha+1}$–measurable uniformization $f\colon \pi_1[A] \to X$ of $A$ and a countable collection $\mathcal{B}$ of pairwise disjoint Borel sets satisfying conditions (i)–(iii) of Lemma 3.2.

Suppose that graph$(f) \notin \mathcal{M}_{\alpha+2}(\mathcal{F},\mathcal{I})$. Then, by the definition of $\mathcal{M}_{\alpha+2}(\mathcal{F},\mathcal{I})$, for every $C \in \Sigma^0_{\alpha+2}$ with $C \supseteq$ graph$(f)$ there is $g \in \mathcal{F}$ such that $\{x\colon g(x) \in C_x\} \notin \mathcal{I}$. Since graph$(f) \in \Sigma^0_{\alpha+2}$, there is $g \in \mathcal{F}$ with

$$\{x\colon g(x) \in \mathrm{graph}(f)_x\} = \{x\colon g(x) = f(x)\} \notin \mathcal{I}.$$

Let $B' = \{x\colon g(x) = f(x)\}$. There is $B \in \mathcal{B}$ with $B' \cap B \notin \mathcal{I}$. Since $f|B = (y_B \cdot h)|B$, we have $g|(B' \cap B) = (y_B \cdot h)|(B' \cap B)$, and therefore $(y_B^{-1} \cdot g)|(B' \cap B) = h|(B' \cap B)$. As $\mathcal{F}$ is left shift invariant, $y_B^{-1} \cdot g \in \mathcal{F}$, but this contradicts our assumption on $h$. Hence, graph$(f) \in \mathcal{M}_{\alpha+2}(\mathcal{F},\mathcal{I})$.

By the choice of $A$ for every $M \in \mathcal{M}_\alpha(\mathcal{I})$ there is $x \in \pi_1[A]$ such that $f(x) \notin M_x$. So, the graph of $f$ is in $\mathcal{M}_{\alpha+2}(\mathcal{F},\mathcal{I}) \setminus \mathcal{M}_\alpha(\mathcal{I})$. ∎

Since $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) \subseteq \mathcal{M}_\alpha(\mathcal{I})$ for every $3 \leq \alpha < \omega_1$, the following corollary holds.

**Corollary 3.4** *Consider $\mathcal{I}$ and $\mathcal{F}$ as in the above theorem. Then for every $3 \leq \alpha < \omega_1$ we have $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+2}(\mathcal{F},\mathcal{I})$. In particular, $\mathcal{M}_\alpha(\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+2}(\mathcal{I})$.*

**Corollary 3.5** *If $\mathcal{J}$ is one of the following ideals of subsets of $\mathbb{R}^2$, then $\mathcal{J}_\alpha \subsetneq \mathcal{J}_{\alpha+2}$ for every $3 \leq \alpha < \omega_1$:*

*(1) the ideal of sets $A$ such that $A$ can be covered by a Borel set whose all vertical sections are null (meager);*

*(2) the ideal of sets $A$ such that $A$ can be covered by a Borel set whose all vertical and horizontal sections are null (meager);*

*(3) the ideal of sets $A$ such that $A$ can be covered by a Borel set which is null (meager) in every direction;*

*(4) the ideal of sets $A$ such that $A$ can be covered by a Borel set which is null (meager) on vertical sections and on graphs of polynomials.*

**Proof.** Consider $X = \mathbb{R}$ with addition. Put $\mathcal{F}_{(1)} = \emptyset$, let $\mathcal{F}_{(2)}$ be the family of constant functions, $\mathcal{F}_{(3)}$ - the family of linear mappings and $\mathcal{F}_{(4)}$ - the family of polynomials. Then $\mathcal{J} = \mathcal{M}(\mathcal{F}_{(i)}, \mathsf{Null})$ (or $\mathcal{J} = \mathcal{M}(\mathcal{F}_{(i)}, \mathsf{Meager})$). The result follows from Corollary 3.4. ∎

Note that in fact, one can prove that $\mathcal{M}_\alpha(\mathcal{J}) \subsetneq \mathcal{M}_{\alpha+2}(\mathcal{J})$ assuming only that $\mathcal{J}$ is a $\sigma$–ideal with a Borel base, containing all singletons, and which is $\Sigma^0_\alpha$-on-$\Pi^0_\alpha$ and $\Sigma^0_{\alpha+1}$-on-$\Pi^0_{\alpha+1}$. To see this, use Theorem 2.1 to find a $\Sigma^0_{\alpha+1}$–measurable unformization $f$ of $\Pi^0_\alpha$ set $A$ from Theorem 2.2, and note that graph$(f) \in \mathcal{M}_{\alpha+2}(\mathcal{J}) \setminus \mathcal{M}_\alpha(\mathcal{J})$.

# 4 Fubini products

Notice that $\mathcal{M}(\mathcal{I})$ can be seen as $\{\emptyset\} \otimes \mathcal{I}$. Fubini products of null or meager ideals have bases of bounded Borel complexity (e.g. $\mathsf{Null} \otimes \mathsf{Null}$, $\mathsf{Null} \otimes \mathsf{Meager}$, see e.g. [2]), so we cannot replace $\{\emptyset\}$ by $\mathsf{Null}$ or $\mathsf{Meager}$ if we want to obtain a $\sigma$–ideal with the complex Borel base property. However, we will show that a $\sigma$–ideal of the form $\mathcal{J} \otimes \mathsf{Null}$ ($\mathcal{J} \otimes \mathsf{Meager}$) has the complex Borel base property if $\mathcal{J}$ has property (M) and a Borel base.

**Definition 4.1** *We will say that an ideal $\mathcal{J}$ of subsets of a Polish space $X$ has* property (M) *if there is a Borel function $f\colon X \to [0,1]$ such that $f^{-1}[\{x\}] \notin \mathcal{J}$ for every $x \in [0,1]$.*

Notice that every uncountable Polish space $X$ can play the role of $[0,1]$ in the above theorem, since such $X$ is Borel isomorphic to $[0,1]$.
Let $X$ be an uncountable Polish space and let $C \subseteq X$ be a set homeomorphic with the Cantor space $\{0,1\}^\omega$. Since $\{0,1\}^\omega$ is homeomorphic with $\{0,1\}^\omega \times \{0,1\}^\omega$, there is a continuous bijection $g\colon C \to \{0,1\}^\omega \times \{0,1\}^\omega$ and for every $x \in \{0,1\}^\omega$ the preimage $g^{-1}[\{0,1\}^\omega \times \{x\}]$ is uncountable. Then $f = \pi_2 \circ g$ witnesses that $\sigma$-ideal of countable subsets has property (M). Using a similar argument and the Silver Theorem [11, 35.20] one can show that ideals generated by an coanalytic equivalence relation with uncountable many equivalence classes (i.e. ideals of sets which can be covered by countably many equivalence classes) have property (M). By [7] the same holds for the ideal of subsets of the plane which can be covered by countably many lines. There are other $\sigma$–ideals with property (M): ideals on Polish spaces with a $\Sigma^0_2$ base which are not ccc (see [13]), some ideals defined by translations (see [3] and [14]). For the further discussion on property (M) see [1], [3] and [4].

**Theorem 4.2** *Let $\mathcal{I}$ be a $\sigma$–ideal of subsets of an uncountable Polish space $X$. Suppose $\mathcal{I}$ has a Borel base, is $\Sigma^0_\alpha$-on-$\Pi^0_\alpha$ for each $\alpha < \omega_1$ and contains all singletons. If a $\sigma$–ideal $\mathcal{J}$ of subsets of $X$ has property (M) then there is $\beta < \omega_1$ such that $(\mathcal{J}\otimes\mathcal{I})_\alpha \subsetneq (\mathcal{J}\otimes\mathcal{I})_{\alpha+2}$ for each $\alpha > \beta$.*

**Proof.** Let $f\colon X \to X$ be a Borel function witnessing (M) for $\mathcal{J}$, i.e. such that

$$f^{-1}[\{x\}] \notin \mathcal{J} \text{ for every } x \in X.$$

Let $\gamma$ be such that $f$ is $\Sigma^0_\gamma$–measurable. Fix $\gamma < \beta < \omega_1$ such that $\gamma + \beta = \beta$ (e.g. $\beta = \gamma \cdot \omega$) and notice that if $\alpha > \beta$ then $\gamma + \alpha = \alpha$. Let $\beta \leq \alpha < \omega_1$ and let $U \subseteq X \times X^2$ be universal for $\Sigma^0_\alpha(X^2)$. Define $H\colon X^3 \to X^3$ by $H(x,y,z) = (f(x),y,z)$ and let $V = H^{-1}[U]$. Clearly $V \in \Sigma^0_{\gamma+\alpha} = \Sigma^0_\alpha$. Consider $A = \{(x,y) \in X^2\colon (x,x,y) \notin V\}$, a $\Pi^0_\alpha$ set.
Since $\mathcal{I}$ is $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$, the set $A' = A \cap (\{x\colon A^c_x \in \mathcal{I}\} \times X)$ is also $\Pi^0_\alpha$. By Theorem 2.1, the set $A'$ has a uniformization with a graph $C$ of class $\Pi^0_{\alpha+1}$. Clearly $C \in \mathcal{M}_{\alpha+2}(\mathcal{I})$.
We will show that $C \notin (\mathcal{J}\otimes\mathcal{I})_\alpha$ by proving that $C \nsubseteq E$ for $E \in \Sigma^0_\alpha(X^2) \cap (\mathcal{J}\otimes\mathcal{I})$. Of course $E = U_x$ for some $x \in X$. So, $E = V_t$ for each $t \in f^{-1}[\{x\}]$. Notice that $C_t \cap E_t = \emptyset$ since $E_t = (V_t)_t = V_{(t,t)} = X \setminus A_t$. By the definition of Fubini product $\{t \in f^{-1}[\{x\}]\colon E_t \notin \mathcal{I}\} \in \mathcal{J}$. As $f^{-1}[\{x\}] \notin \mathcal{J}$, we have also that $\{t \in f^{-1}[\{x\}]\colon E_t \in \mathcal{I}\} \notin \mathcal{J}$. Moreover, if $t \in f^{-1}[\{x\}]$ and $E_t \in \mathcal{I}$ then $A^c_t = E_t \in \mathcal{I}$. Hence there is $t$ such that $(A'_t) \cap E_t = \emptyset$ and $A'_t \supseteq C_t \neq \emptyset$. Therefore $C \setminus E \neq \emptyset$, which means that $C \notin (\mathcal{J}\otimes\mathcal{I})_\alpha$. Since $\mathcal{M}_{\alpha+2}(\mathcal{I}) \subseteq (\mathcal{J}\otimes\mathcal{I})_{\alpha+2}$, it follows that $C \in (\mathcal{J}\otimes\mathcal{I})_{\alpha+2} \setminus (\mathcal{J}\otimes\mathcal{I})_\alpha$.

■

In [6], the $\sigma$-ideals $\mathcal{M}(\mathsf{Null})$ and $\mathcal{M}(\mathsf{Meager})$ are called Mokobodzki $\sigma$-ideals. The results presented here and in the previous section indicate how we can generalize the definition of a Mokobodzki ideal.

Let $(X, \cdot)$ be an uncountable Polish group. We say that $Y \subseteq X^2$ belongs to a $\sigma$–ideal $\mathcal{M}(\mathcal{F}, \mathcal{I}, \mathcal{J})$ on $X^2$ if $Y \in \mathcal{J} \otimes \mathcal{I}$ and $Y$ can be covered by a Borel set $B \subseteq X^2$ such that

$$\{y \in X \colon \{x \in X \colon (x, y \cdot f(x)) \in B\} \notin \mathcal{I}\} \in \mathcal{J}$$

for every $f \in \mathcal{F}$. We say that a $\sigma$–ideal is a *generalized Mokobodzki ideal* if it is of the form $\mathcal{M}(\mathcal{F}, \mathcal{I}, \mathcal{J})$ for $\mathcal{I} = \mathsf{Null}$ or $\mathcal{I} = \mathsf{Meager}$, a family of Borel functions $\mathcal{F} \subseteq X^X$ containing constants which is left shift invariant and is not $\mathcal{I}$–ubiquitous, and a $\sigma$–ideal $\mathcal{J}$ with property (M). Loosely speaking, a generalized Mokobodzki ideal is a $\sigma$–ideal consisting of sets $C$ which are null (meager) along all functions from $\mathcal{F}$ and which have null (meager) vertical section $C_x$ for $\mathcal{J}$–almost every $x$. By combining proofs of Theorem 3.3 and Theorem 4.2 it is possible to show that all generalized Mokobodzki ideals have the complex Borel base property. We decided to prove it in less general cases for the sake of clarity.

# 5 Set-theoretic properties

In this section we explore some properties of the ideals considered in the previous sections. First, we show that they have property (M). As before, throughout this section we assume that $X$ is an uncountable Polish space.

**Proposition 5.1** *If $\mathcal{I}$ is a $\sigma$–ideal of subsets of $X$ such that $X \notin \mathcal{I}$ and $\mathcal{F} \subseteq X^X$, then $\mathcal{M}(\mathcal{F}, \mathcal{I})$ has property (M). Also, if a $\sigma$–ideal $\mathcal{J}$ has property (M), then $\mathcal{J} \otimes \mathcal{I}$ has property (M).*

**Proof.** Consider the function $\pi_1(x, y) = x$. Then $\pi_1^{-1}[\{x\}] = \{x\} \times X \notin \mathcal{M}(\mathcal{F}, \mathcal{I})$. To prove the second part of the theorem consider $g = f \circ \pi_1$, where $f \colon X \to X$ witnesses that $\mathcal{J}$ has property (M).

■

Notice, that this means that the ideals from the previous sections are very far from being ccc. Notice also that the above proposition together with Theorem 4.2 allows us to produce easily a lot of examples of $\sigma$–ideals with the complex Borel base property. In particular, Fubini products of $\mathsf{Null}$ or $\mathsf{Meager}$ ideals with ideals from Proposition 5.1 have the complex Borel base property.

In [6, 2.4] it was proved that if $\mathcal{I}$ is a proper $\sigma$–ideal of subsets of $X$, then $\mathcal{M}(\mathcal{I})$ has additivity $\omega_1$. The proof uses Theorem 2.2, Kondo–Adisson and Sierpiński theorems.

In fact, it seems that the essential reason for the equality $\mathrm{add}(\mathcal{M}(\mathcal{I})) = \omega_1$ lies in the fact that $\mathcal{M}(\mathcal{I})$ is an union of sequences of $\omega_1$ strictly increasing $\sigma$–ideals, $\mathcal{I} \in \{\mathsf{Null}, \mathsf{Meager}\}$. More generally, the following fact holds.

**Proposition 5.2** *Suppose that $\mathcal{I}$ has a Borel base and there are $\sigma$–ideals $\mathcal{I}_\alpha$ with $\mathcal{I} = \bigcup_{\alpha<\kappa} \mathcal{I}_\alpha$ and $\mathcal{I}_\alpha \subsetneq \mathcal{I}_{\alpha+1}$ for every $\alpha < \kappa$. Then $\mathrm{add}(\mathcal{I}) \leq \kappa$. In particular, if $\mathcal{I}$ has the complex Borel base property, then $\mathrm{add}(\mathcal{I}) = \omega_1$.*

**Proof.** For every $\alpha < \kappa$ take $A_\alpha \in \mathcal{I}_{\alpha+1} \setminus \mathcal{I}_\alpha$ and consider a set $A = \bigcup_{\alpha<\kappa} A_\alpha$. If $A$ is in $\mathcal{I}$, then we could find Borel $B \in \mathcal{I}$ with $B \supseteq A$, but there would be $\alpha < \kappa$ with $B \in \mathcal{I}_\alpha$. This would contradict the fact that $A_\alpha \subseteq B$ and $A_\alpha \notin \mathcal{I}_\alpha$.

The second part of the statement follows easily.

■

As a corollary we obtain that the ideals from Theorem 4.2 and Theorem 3.3 (and, all generalized Mokobodzki ideals) have additivity $\omega_1$. In fact, every $\sigma$–ideal containing all singletons is a union of a strictly increasing sequence of its proper sub–$\sigma$–ideals.

Indeed, let $\mathcal{I}$ be such an ideal. Define by $\mathrm{cof}_\sigma(\mathcal{I})$ the minimal cardinality of a family $\mathcal{A} \subseteq \mathcal{I}$ such that $\mathcal{A}$ $\sigma$–generates $\mathcal{I}$. Clearly, $\omega < \mathrm{cof}_\sigma(\mathcal{I}) \leq \mathrm{cof}(\mathcal{I})$. Let $\{B_\xi \colon \xi < \mathrm{cof}_\sigma(\mathcal{I})\}$ be a family witnessing $\mathrm{cof}_\sigma(\mathcal{I})$ and let $\mathcal{I}_\xi$ be the $\sigma$–ideal generated by $\{B_\alpha \colon \alpha < \xi\}$. Notice that the sequence $(\mathcal{I}_\xi)_{\xi < \mathrm{cof}_\sigma(\mathcal{I})}$ is increasing and does not stabilise. Let $\mathrm{cofin}(\mathcal{I}) = \min\{\kappa \colon \mathcal{I} = \bigcup_{\alpha<\kappa} \mathcal{I}_\alpha$ where $(\mathcal{I}_\alpha)$ is a strictly increasing sequence of $\sigma$–ideals which are proper subideals of $\mathcal{I}\}$. It is well–defined by the remark above. Clearly $\mathrm{cofin}(\mathcal{I})$ is regular (and is not smaller than the cofinality of $\mathrm{cof}_\sigma(\mathcal{I})$). Moreover, an argument as in the proof of Proposition 5.2 implies that $\mathrm{add}(\mathcal{I}) \leq \mathrm{cofin}(\mathcal{I})$. If $\mathcal{I}$ has the complex base property then $\mathrm{cofin}(\mathcal{I}) = \omega_1$. It is interesting how this coefficient behaves in other situations. In the last section we pose one of the natural questions in this context.

We will turn now to classical ideal invariants. Some cardinal coefficients of $\mathcal{M}(\mathcal{F},\mathcal{I})$ are inherited from the ideal $\mathcal{I}$.

**Proposition 5.3** *If $\mathcal{I}$ is a $\sigma$–ideal of subsets of $X$, $\mathcal{I}$ has a Borel base, and $\mathcal{F} \subseteq X^X$, then*

*(i)* $\mathrm{cov}(\mathcal{M}(\mathcal{F},\mathcal{I})) = \mathrm{cov}(\mathcal{I})$*;*

*(ii)* $\mathrm{non}(\mathcal{M}(\mathcal{F},\mathcal{I})) = \mathrm{non}(\mathcal{I})$.

**Proof.** Suppose that the family $\{A_\xi \colon \xi < \kappa\} \subseteq \mathcal{I}$ covers $X$. We may assume that each $A_\xi$ is Borel. Then $\{A_\xi \times A_\eta \colon \xi, \eta < \kappa\} \subseteq \mathcal{M}(\mathcal{F},\mathcal{I})$ and it covers $X^2$. On the other hand if $\{Z_\xi \colon \xi < \kappa\} \subseteq \mathcal{M}(\mathcal{F},\mathcal{I})$ covers $X^2$ and $x \in X$, then $\{(Z_\xi)_x \colon \xi < \kappa\} \subseteq \mathcal{I}$ and it covers $X$.

If $Z \notin \mathcal{I}$, then $\{x\} \times Z \notin \mathcal{M}(\mathcal{F},\mathcal{I})$ and $|\{x\} \times Z| = |Z|$, where $x \in X$. On the other hand, if $|Z| < \mathrm{non}(\mathcal{I})$ for $Z \subseteq X^2$, then $|\pi_1[Z]| < \mathrm{non}(\mathcal{I})$ and $|\pi_2[Z]| < \mathrm{non}(\mathcal{I})$. So, there is a Borel $B \in \mathcal{I}$ such that $\pi_1[Z] \cup \pi_2[Z] \subseteq B$. Therefore, $Z \subseteq B \times B$ and $Z \in \mathcal{M}(\mathcal{F},\mathcal{I})$.

■

The assumption that $\mathcal{I}$ has a Borel base in part (i) of the assertion of Proposition 5.3 is needed. To see this, let $\{B_\xi \colon \xi < \omega_1\}$ be a collection of pairwise disjoint Bernstein sets in $[0,1]$ which covers $[0,1]$. Define $\mathcal{I}$ as the $\sigma$-ideal of subsets of $[0,1]$ which can be covered by countably many sets $B_\xi$. Then $\mathrm{cov}(\mathcal{I}) = \omega_1$. But $\mathcal{M}(\mathcal{I})$ consists of sets with countable sections, hence $\mathrm{cov}(\mathcal{M}(\mathcal{I})) = \mathfrak{c}$ and consistently $\mathrm{cov}(\mathcal{M}(\mathcal{I})) > \mathrm{cov}(\mathcal{I})$.

In the case of ideals defined in Section 4 we can apply the following fact.

**Proposition 5.4 (folklore)** *If $\mathcal{I}$ and $\mathcal{J}$ are $\sigma$–ideals of subsets of $X$ with Borel bases then*

$$\mathrm{non}(\mathcal{J} \otimes \mathcal{I}) = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\} \text{ and}$$

$$\mathrm{cov}(\mathcal{J} \otimes \mathcal{I}) = \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}.$$

**Proof.** Let $Z$, $Y$ witness $\mathrm{non}(\mathcal{J})$ and $\mathrm{non}(\mathcal{I})$ respectively. Then $Z \times Y \notin \mathcal{J} \otimes \mathcal{I}$ and $|Z \times Y| = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\}$. If $Z$ witnesses $\mathrm{non}(\mathcal{J} \otimes \mathcal{I})$ then $|\pi_1[Z]| \geq \mathrm{non}(\mathcal{J})$ and $|\pi_2[Z]| \geq \mathrm{non}(\mathcal{I})$. Therefore, $\mathrm{non}(\mathcal{J} \otimes \mathcal{I}) = \max\{\mathrm{non}(\mathcal{I}), \mathrm{non}(\mathcal{J})\}$.
Assume that $\{A_\xi \colon \xi < \kappa\} \subseteq \mathcal{J}$ covers $X$. Then, $\{A_\xi \times X \colon \xi < \kappa\} \subseteq \mathcal{J} \otimes \mathcal{I}$ and it covers $X^2$. A similar argument for $\mathcal{I}$ shows that $\mathrm{cov}(\mathcal{J} \otimes \mathcal{I}) \leq \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}$. On the other hand, if $\mathrm{cov}(\mathcal{J} \otimes \mathcal{I}) \leq \mathrm{cov}(\mathcal{J})$ and $\{Z_\xi \colon \xi < \mathrm{cov}(\mathcal{J} \otimes \mathcal{I})\}$ covers $X^2$, then there is $x \in X$ such that $(Z_\xi)_x \in \mathcal{I}$ for every $\xi < \kappa$. As $(Z_\xi)_x$ covers $X$, we have $\mathrm{cov}(\mathcal{J} \otimes \mathcal{I}) \geq \min\{\mathrm{cov}(\mathcal{I}), \mathrm{cov}(\mathcal{J})\}$.

■

Let $\mathcal{I}$ be a $\sigma$–ideal on a Polish space $X$. Recall that $H \subseteq X$ is *an $\mathcal{I}$–hull* of a set $A \subseteq X$ if $H$ is Borel and for every Borel $B \subseteq H$ either $B \in \mathcal{I}$ or $B \cap A \neq \emptyset$. We say that a $\sigma$–ideal $\mathcal{I}$ on $X$ has *the hull property* if every subset of $X$ has an $\mathcal{I}$–hull.

**Proposition 5.5** *If $\mathcal{I}$ is a Borel–on–Borel $\sigma$–ideal on $X$, then $\mathcal{M}(\mathcal{I})$ does not possess the hull property.*

**Proof.** Consider $A = V \times X$, where $V \subseteq X$ is a Bernstein set. Assume $H \supseteq A$ is a Borel set and consider $L = \{x \in X \colon H_x \notin \mathcal{I}\}$. Notice that $L$ is Borel. Since $V \subseteq L$, the set $L$ has to be co–countable. In particular, there is $x \in L \setminus V$ and the set $\{x\} \times H_x$ is an $\mathcal{M}(\mathcal{I})$–positive Borel subset of $H$ disjoint with $A$.

■

# 6 Additional remarks and open questions

It is well known that every Borel function on $[0,1]^2$ is nice on a big domain (of Baire class 1 on a set of Lebesgue measure 1, continuous on a co–meager set). However, in [2] it is proved that this is not the case if instead of $\mathsf{Null}$ or $\mathsf{Meager}$ ideals we would consider $\mathcal{M}(\mathsf{Null})$ or $\mathcal{M}(\mathsf{Meager})$. More precisely, for every $\alpha \geq 2$ there is a Borel function $g \colon [0,1]^2 \to [0,1]$ such that for every Borel set $M \in \mathcal{M}(\mathsf{Null})$ (or $\mathcal{M}(\mathsf{Meager})$) we can find $x \in [0,1]$ such that the function $g_x | M_x^c$ is not $\Sigma^0_\alpha$-measurable (here $g_x(y) = g(x,y)$). On the other hand (see [2]), these pathological points can be covered by null and meager sets, i.e. for every $M$ as above, the set

$$\{x \colon g_x | M_x^c \text{ is not } \Sigma^0_\alpha\text{-measurable}\} \in \mathsf{Null} \cap \mathsf{Meager}.$$

The following notion is motivated by the above considerations.
Let $X$ be an uncountable Polish space. Let $\mathcal{J}$, $\mathcal{I}$ be $\sigma$–ideals of subsets of $X$. We say that $\mathcal{J}$ is *$\mathcal{I}$–thin* if for each $2 \leq \alpha < \omega_1$ there is a Borel function $g \colon X^2 \to X$ such that for all $M \in \mathcal{M}_\alpha(\mathcal{I})$ and $C \in \mathcal{J}$ there is $t_0 \in X \setminus C$ such that $g_{t_0} | (X \setminus M_{t_0})$ is not $\Sigma^0_\alpha$-measurable. Using the notion of thinness, the theorems cited above can be stated in this form: $\{\emptyset\}$ is $\mathsf{Null}$–thin and $\mathsf{Meager}$–thin, whereas $\mathsf{Null}$ and $\mathsf{Meager}$ are not. We show the following.

**Proposition 6.1** *Every $\sigma$–ideal with property (M) is $\mathcal{I}$–thin for every $\sigma$–ideal $\mathcal{I}$ that is $\Sigma^0_\alpha$–on–$\Pi^0_\alpha$ for all $\alpha > \alpha_0$.*

**Proof.** Let $\mathcal{I}$ be a Borel–on–Borel $\sigma$–ideal on $X$ and let $\mathcal{J}$ be a $\sigma$–ideal on $X$ with the property (M). Assume that $2 \leq \alpha < \omega_1$. Let $f\colon X \to X$ be a Borel function witnessing (M) for $\mathcal{J}$. Let $U \subseteq X \times X^2$ be universal for $\Sigma^0_\alpha(X^2)$. As in the proof of Theorem 4.2 define $H\colon X^3 \to X^3$ by $H(x,y,z) = (f(x),y,z)$, let $V = H^{-1}[U]$ and consider $A = \{(x,y) \in X^2\colon (x,x,y) \notin V\}$. Notice that $A$ is Borel, and thus the set

$$A' = A \cap \{(t,y)\colon A^c_t \in \mathcal{I}\}$$

is $\Sigma^0_\gamma$ for some $\gamma < \omega_1$.
Let $F\colon \{0,1\}^\omega \times X \to X$ be a function from Theorem 2.1 for the set $A'$ and the ordinal $\gamma$. Define $G\colon \{0,1\}^\omega \times X \to X^2$ by

$$G(c,x) = (x, F(c,x)).$$

Consider a Borel subset $D$ of $\{0,1\}^\omega$ such that $D \notin \Sigma^0_\alpha(\{0,1\}^\omega)$ and let

$$\tilde{D} = G[D \times \pi_1[A']].$$

Notice that since $G$ is one–to–one and Borel, the set $\tilde{D}$ is Borel. Clearly $\tilde{D} \subseteq A'$. Let $g\colon X^2 \to \{x_0, x_1\}$ be the characteristic function of $\tilde{D}$, where $x_0$ and $x_1$ are distinct points of $X$. Let $M \in \mathcal{M}_\alpha(\mathcal{I}) \cap \Sigma^0_\alpha$ and $C \in \mathcal{J}$. There is $x \in X$ such that $U_x = M$. So, for every $t \in f^{-1}[\{x\}]$ we have $A_t = M^c_t$ and, therefore, $t \in \pi_1[A']$. Since $f^{-1}[\{x\}] \setminus C \neq \emptyset$, there is $t_0 \in X \setminus C$ with

$$\left(g_{t_0}|(X \setminus M_{t_0})\right)^{-1}[\{x_1\}] = (X \setminus M_{t_0}) \cap \tilde{D}_{t_0} = \tilde{D}_{t_0} = G[D \times \{t_0\}].$$

Since $c \mapsto G(c,t_0)$ is continuous and one-to-one, the set $G[D \times \{t_0\}]$ is homeomorphic to $D$. Hence $g_{t_0}|(X \setminus M_{t_0})$ is not $\Sigma^0_\alpha$-measurable.

■

We finish with a list of open questions.

**Problem 6.2** *Suppose $X$ is an uncountable Polish space and $\mathcal{I}$ is a $\sigma$–ideal of subsets of $X$.*

*(i) Is it true that $\mathcal{M}_\alpha(\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+1}(\mathcal{I})$ or even $\mathcal{M}_\alpha(\mathcal{F},\mathcal{I}) \subsetneq \mathcal{M}_{\alpha+1}(\mathcal{F},\mathcal{I})$ for every $\alpha < \omega_1$, $\mathcal{I} \in \{\mathsf{Null}, \mathsf{Meager}\}$ and every family $\mathcal{F}$ as in Theorem 3.3?*

*(ii) Is it provable in ZFC that for every $\mathcal{I}$ containing all singletons we have* $\mathrm{add}(\mathcal{I}) = \mathrm{cofin}(\mathcal{I})$*?*

*(iii) Does there exist $\mathcal{I}$ with the hull property and the complex Borel base property?*

*(iv) Let $\mathcal{K}([0,1]^2)$ stand for the set of all nonempty compact subsets of* $[0,1]^2$ *with the Vietoris topology. What is the complexity of the set* $\{K \in \mathcal{K}([0,1]^2)\colon K \in \mathcal{M}(\mathcal{I})\}$ *for $\mathcal{I} \in \{\mathsf{Null}, \mathsf{Meager}\}$? Is it $\Pi^1_1$-complete?*

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*Instytut Matematyczny, Uniwersytet Wrocławski*
*plac Grunwaldzki 2/4, 50-384 Wrocław*
PBOROD@MATH.UNI.WROC.PL
`http://www.math.uni.wroc.pl/~pborod`

*Institute of Mathematics, Technical University of Łódź*

*ul. Wólczańska 215, 90-924 Łódź*
szymon2377@o2.pl
`http://im0.p.lodz.pl/~sglab/indexeng.html`