Measures on Bell's space

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joint work with Mirna Džamonja

Measures on Boolean algebras

- We consider finitely–additive measures on Boolean algebras;
- A measure μ is strictly positive on $\mathfrak{A}$ if $\mu(A) > 0$ for each $A \in \mathfrak{A}^+$. In this case we say that $\mathfrak{A}$ *supports* μ;
- Every (finitely–additive) measure on $\mathfrak{A}$ can be uniquely extended to a (σ–additive) Radon measure on $\mathrm{Stone}(\mathfrak{A})$.

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Separable measures

Definition

A measure μ on a Boolean algebra $\mathfrak{A}$ is *separable* if there is a countable $\mathcal{B} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(A \triangle B) \colon B \in \mathcal{B}\} = 0$$

for each $A \in \mathfrak{A}$

Equivalently...

A measure μ on $\mathfrak{A}$ is separable iff

- the (pseudo–)metric space $(\mathfrak{A}, d_\mu)$ is separable, $d_\mu(A, B) = \mu(A \triangle B)$
- the space $L_1(\mu)$ is separable.

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Spaces with small measures

Problem

How to characterize Boolean algebras carrying only separable measures?

Theorem (Fremlin)

Under MA(ω_1) a Boolean algebra $\mathfrak{A}$ carries a non–separable measure if and only if $\mathfrak{A}$ contains an uncountable independent sequence.

In ZFC: ?

(This is one of the problems connected to the programme of the classification of finitely–additive measures.)

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Uniformly regular measures

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A measure μ on $\mathfrak{A}$ is *uniformly regular* if there is a countable family $\mathcal{D} \subseteq \mathfrak{A}$ such that

$$\inf\{\mu(A \setminus D) \colon D \in \mathcal{D},\ D \subseteq A\} = 0$$

for every $A \in \mathfrak{A}$.

Equivalently. . .

A measure μ on $\mathfrak{A}$ is uniformly regular if and only if

- μ is a G_δ point in the space of probability Radon measures on $\mathrm{Stone}(\mathfrak{A})$ with weak* topology (Pol, 1982).

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Characterization of uniform regularity

Theorem (Džamonja, Pbn)

If a Boolean algebra supports a non–atomic uniformly regular measure, then it is isomorphic to a subalgebra of the Jordan algebra containing a dense Cantor subalgebra.

Separability versus uniform regularity

The obvious connection:

Remark

Every uniformly regular measure is separable.

Less obvious connection:

Theorem (Plebanek, PBN)

All Boolean algebras without a non–separable measure carry a uniformly regular measure.

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Can we prove an analogous theorem for strictly positive measures? I.e. is it true that all Boolean algebras supporting a measure either supports a uniformly regular measure or a non–separable one?

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Bell's space

Theorem (Bell)

There is a zero–dimensional compact separable space K without a countable π–base and which cannot be mapped continuously onto $[0,1]^{\omega_1}$.

- K is compact zerodimensional, so $\mathfrak{A} = \mathrm{Clopen}(K)$ is a Boolean algebra;
- K is separable, so $\mathfrak{A}$ supports a measure;
- K has no countable π–base, so it does not support a uniformly regular measure;
- is every measure on K separable?

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Construction

For $A \subseteq \omega$ let

$$A^0 = \{x \in 2^\omega : x(n) = 0 \text{ for each } n \in A\}$$

For $\mathcal{A} \subseteq P(\omega)$ let

$$\mathcal{A}^0 = \{A^0 : A \in \mathcal{A}\}.$$

Let $K(\mathcal{A})$ be

$$K(\mathcal{A}) = \mathrm{Stone}(\text{algebra generated by } \mathcal{A}^0) \subseteq P(2^\omega).$$

Example: $K(\mathrm{Fin}) = 2^\omega$.

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Separability

Fact

$K(\mathcal{A})$ is separable for each $\mathcal{A} \subseteq P(\omega)$.

- consider $g \in 2^\omega$;
- let
$$\mathcal{F}_g = \{B \in \mathrm{alg}(\mathcal{A}^0) \colon g \in B\};$$
- it is a filter on $\mathrm{alg}(\mathcal{A}^0)$;
- let $x_g \in K(\mathcal{A})$ be any ultrafilter extending $\mathcal{F}_g$;
- the set
$$\{x_g \colon \mathrm{supp}(g) \text{ is finite}\}.$$
is dense in $K(\mathcal{A})$.

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Cofinality and π-bases

Fact

If $\mathcal{A}$ does not contain a countable cofinite subfamily, then $K(\mathcal{A})$ does not have a countable π–base.

Independence

Remark

If $\mathcal{A}$ contains an uncountable almost disjoint family, then $\mathrm{alg}(\mathcal{A}^0)$ contains an uncountable independent sequence.

- let $\{A_\alpha \colon \alpha < \omega_1\} \subseteq \mathcal{A}$ be almost disjoint ($A_\alpha \cap A_\beta$ is finite for each $\alpha \neq \beta$);
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The space

Theorem (Džamonja, PBN)

Let $\{T_\alpha \colon \alpha < \omega_1\} \subseteq P(\omega)$ be such that for each $\alpha < \beta < \omega_1$

- $T_0 = \emptyset$,
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Let

$$\mathcal{T} = \{T \colon T =^* T_\alpha \text{ for some } \alpha < \omega_1\}.$$

Then $K(\mathcal{T})$ supports only separable measures. Consequently, the Boolean algebra $\mathrm{alg}(\mathcal{T}^0)$ supports only separable measures but not a uniformly regular one.

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The end

Thank you for your attention!

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