

Small compact spaces and convergent sequences

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- 1 Compact spaces and Banach spaces - Problems
 - Efimov problem
 - Mazur vs. Gelfand–Phillips property
- 2 Boolean algebras - Tools
 - Stone isomorphism
 - Minimally generated Boolean algebras
 - Efimov problem
 - Mazur vs. Gelfand–Phillips property
- 3 Set Theory

Efimov problem

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Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under \diamond (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $\text{cof}([s]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

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Efimov problem and measures

Separable measures

A measure μ on a compact space K is *separable* if the underlying space $L_1(\mu)$ is separable.

- if K contains a copy of $\beta\omega$, then there is a continuous surjection $f: K \rightarrow \{0, 1\}^{\mathfrak{c}}$;
- if there is a continuous surjection $f: K \rightarrow \{0, 1\}^{\omega_1}$, then K admits a non-separable measure.

Stronger version of Efimov Problem

Is there an infinite compact space X such that

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Gelfand–Phillips Property and Mazur property

Mazur property

A Banach space X has a *Mazur property* if every *weak**-sequentially continuous $x^{**} \in X^{**}$ is continuous.

A bounded subset A of a Banach space X is said to be *limited* if

$$\lim_{n \rightarrow \infty} \sup_{x \in A} |x_n^*(x)| = 0$$

for every *weak**-null sequence $x_n^* \in X^*$.

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A Banach space X has a *Gelfand–Phillips property* if every relatively norm compact space is limited.

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Mazur vs. Gelfand–Phillips

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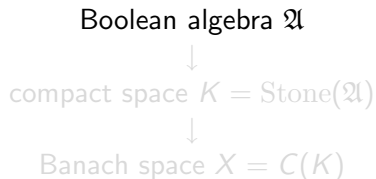
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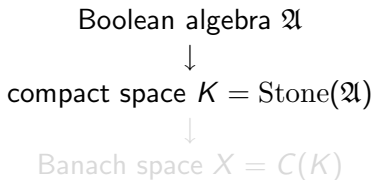
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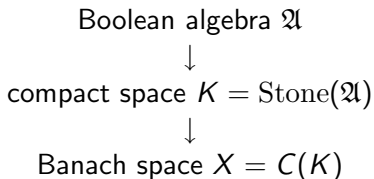
Examples via Stone isomorphism



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Minimally generated Boolean algebras

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A Boolean algebra \mathfrak{A} is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well-ordered.

Minimally generated compact spaces

A compact zero-dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

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Efimov problem and m.g. BA

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Minimally generated B.A. & Efimov problem

Most of the constructions of Efimov spaces mentioned above was minimally generated.

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Is there an infinite compact space X such that

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Measures on minimally generated compact spaces

Theorem (PBN)

Minimally generated compact spaces carry only separable measures.

Conclusions:

- under CH there is a Efimov space admitting only separable measures;
- there is a compactification of ω admitting only separable measures, and without a convergent subsequence of ω ;
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More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K . For $A \subseteq K$ let

$$\text{conv}A = \text{conv}\{\delta_a : a \in A\},$$

i.e. $\text{conv}A$ is the set of all probability measures supported by a finite subset of A . Moreover, put

$$S(A) = \bigcup_{\alpha < \omega_1} \text{scl}_\alpha(A),$$

where $\text{scl}_0(A) = \text{conv}A$ and $\text{scl}_{\alpha+1}(A)$ is the sequential closure of $\text{scl}_\alpha(A)$ in the weak* topology of $P(K)$.

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Mazur property

Theorem (Plebanek)

Let K be a compactification of ω , and suppose that

- $P(K) = S(K)$;
- for every $x \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$, then $\delta_t \in S(Y)$.

Then $C(K)$ has the Mazur property.

Theorem (Plebanek, PBN)

If K is minimally generated, then $P(K) = S(K)$.

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Gelfand–Phillips property

Theorem (Schlumprecht)

If \mathfrak{A}/Fin is dense in $P(\omega)/Fin$, then $C(\text{Stone}(\mathfrak{A}))$ does not have Gelfand-Phillips property.

Theorem (Balcar, Simon, Pelant)

There is a minimally generated Boolean algebra \mathfrak{A} with the above property.

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Mazur vs. Gelfand-Phillips property

Theorem (Plebanek, PBN)

If $\aleph > \mathfrak{h}$, then there is a Banach space satisfying Mazur property but without Gelfand-Phillips property.

Combinatorics

- \mathfrak{p} - pseudo–intersection number;
- \mathfrak{s} - splitting number;
- \mathfrak{h} - distributive number;
- \mathfrak{k} - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f: \omega \rightarrow \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

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Another example of consistency result

Theorem (PBN)

If $\mathfrak{p} = \mathfrak{c}$, then there is a space without a copy of $\beta\omega$ such that every convergent sequence consists of infinitely many points of character \mathfrak{c} .