Small compact spaces and convergent sequences

Piotr Borodulin–Nadzieja

Jamex & Topology and its Applications, Kyoto

4 December 2007

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X does not contain a copy of $\beta\omega$?

A space with the above properties is called a Efimov space.

There are some Efimov spaces, eg.:

- under CH (Fedorchuk, 1976);
- under $\diamondsuit$ (Koppelberg, 1988);
- under $2^{\mathfrak{s}} < 2^{\mathfrak{c}}$ & $cof([\mathfrak{s}]^{\omega}) = \mathfrak{s}$ (Dow, 2005).

Efimov problem and measures

Separable measures

A measure μ on a compact space K is *separable* if the underlying space $L_1(\mu)$ is separable.

- if K contains a copy of $\beta\omega$, then there is a continuous surjection $f: K \to \{0,1\}^{\mathfrak{c}}$;
- if there is a continuous surjection $f: K \to \{0,1\}^{\omega_1}$, then K admits a non–separable measure.

Stronger version of Efimov Problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X admits only separable measures?

Efimov problem and measures

Separable measures

A measure μ on a compact space K is *separable* if the underlying space $L_1(\mu)$ is separable.

- if K contains a copy of $\beta\omega$, then there is a continuous surjection $f\colon K \to \{0,1\}^{\mathfrak{c}}$;
- if there is a continuous surjection $f\colon K \to \{0,1\}^{\omega_1}$, then K admits a non–separable measure.

Stronger version of Efimov Problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X admits only separable measures?

Efimov problem and measures

Separable measures

A measure μ on a compact space K is *separable* if the underlying space $L_1(\mu)$ is separable.

- if K contains a copy of $\beta\omega$, then there is a continuous surjection $f\colon K \to \{0,1\}^{\mathfrak{c}}$;
- if there is a continuous surjection $f\colon K \to \{0,1\}^{\omega_1}$, then K admits a non–separable measure.

Stronger version of Efimov Problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X admits only separable measures?

Efimov problem and measures

Separable measures

A measure μ on a compact space K is *separable* if the underlying space $L_1(\mu)$ is separable.

- if K contains a copy of $\beta\omega$, then there is a continuous surjection $f\colon K \to \{0,1\}^{\mathfrak{c}}$;
- if there is a continuous surjection $f\colon K \to \{0,1\}^{\omega_1}$, then K admits a non–separable measure.

Stronger version of Efimov Problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X admits only separable measures?

Gelfand–Phillips Property and Mazur property

Mazur property

A Banach space X has a *Mazur property* if every *weak**–sequentially continuous $x^{**} \in X^{**}$ is continuous.

A bounded subset A of a Banach space X is said to be *limited* if

$$\lim_{n \to \infty} \sup_{x \in A} |x_n^*(x)| = 0$$

for every weak*–null sequence $x_n^* \in X^*$.

Gelfand–Phillips property

A Banach space X has a *Gelfand–Phillips property* if every relatively norm compact space is limited.

Gelfand–Phillips Property and Mazur property

Mazur property

A Banach space X has a *Mazur property* if every *weak**–sequentially continuous $x^{**} \in X^{**}$ is continuous.

A bounded subset A of a Banach space X is said to be *limited* if

$$\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0$$

for every weak*–null sequence $x_n^* \in X^*$.

Gelfand–Phillips property

A Banach space X has a *Gelfand–Phillips property* if every relatively norm compact space is limited.

Gelfand–Phillips Property and Mazur property

Mazur property

A Banach space X has a *Mazur property* if every *weak**–sequentially continuous $x^{**} \in X^{**}$ is continuous.

A bounded subset A of a Banach space X is said to be *limited* if

$$\lim_{n\to\infty} \sup_{x\in A} |x_n^*(x)| = 0$$

for every weak*–null sequence $x_n^* \in X^*$.

Gelfand–Phillips property

A Banach space X has a *Gelfand–Phillips property* if every relatively norm compact space is limited.

Mazur Property vs. Gelfand-Phillips Property

It is known that there are Banach spaces with a Gelfand–Phillips property but without a Mazur property.

Mazur vs. Gelfand–Phillips

Does Mazur property imply Gelfand–Phillips property?

Mazur Property vs. Gelfand-Phillips Property

It is known that there are Banach spaces with a Gelfand–Phillips property but without a Mazur property.

Mazur vs. Gelfand–Phillips

Does Mazur property imply Gelfand–Phillips property?

Examples via Stone isomorphism

Boolean algebra $\mathfrak{A}$

$\downarrow$

compact space $K = \mathrm{Stone}(\mathfrak{A})$

$\downarrow$

Banach space $X = C(K)$

Examples via Stone isomorphism

Boolean algebra $\mathfrak{A}$

$\downarrow$

compact space $K = \mathrm{Stone}(\mathfrak{A})$

$\downarrow$

Banach space $X = C(K)$

Examples via Stone isomorphism

Boolean algebra $\mathfrak{A}$

$$\downarrow$$

compact space $K = \mathrm{Stone}(\mathfrak{A})$

$$\downarrow$$

Banach space $X = C(K)$

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces,

monotonically normal spaces.

Minimally generated Boolean algebras

Minimally generated Boolean algebras

A Boolean algebra $\mathfrak{A}$ is *minimally generated* if there is a maximal chain in the lattice of its subalgebras and this chain is well–ordered.

Minimally generated compact spaces

A compact zero–dimensional space K is *minimally generated* if it is a Stone space of a minimally generated Boolean algebra.

Examples: metrizable spaces, ordered spaces, scattered spaces, monotonically normal spaces.

Efimov problem and m.g. BA

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X **does not contain a copy of** $\beta\omega$?

Minimally generated B.A. & Efimov problem

Most of the constructions of Efimov spaces mentioned above was minimally generated.

Efimov problem and m.g. BA

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X **does not contain a copy of** $\beta\omega$?

Minimally generated B.A. & Efimov problem

Most of the constructions of Efimov spaces mentioned above was minimally generated.

Efimov problem and m.g. BA

Efimov problem

Is there an infinite compact space X such that

- X does not contain nontrivial convergent sequences,
- X **admits only separable measures**?

Minimally generated B.A. & Efimov problem

Most of the constructions of Efimov spaces mentioned above was minimally generated.

Measures on minimally generated compact spaces

Theorem (PBN)

Minimally generated compact spaces carry only separable measures.

Conclusions:

- under CH there is a Efimov space admitting only separable measures;
- there is a compactification of ω admitting only separable measures, and without a convergent subsequence of ω;
- there is a compact but not sequentially compact space admitting only separable measures (Haydon's theorem).

Measures on minimally generated compact spaces

Theorem (PBN)

Minimally generated compact spaces carry only separable measures.

Conclusions:

- under *CH* there is a Efimov space admitting only separable measures;
- there is a compactification of ω admitting only separable measures, and without a convergent subsequence of ω;
- there is a compact but not sequentially compact space admitting only separable measures (Haydon's theorem).

Measures on minimally generated compact spaces

Theorem (PBN)

Minimally generated compact spaces carry only separable measures.

Conclusions:

- under *CH* there is a Efimov space admitting only separable measures;
- there is a compactification of ω admitting only separable measures, and without a convergent subsequence of ω;
- there is a compact but not sequentially compact space admitting only separable measures (Haydon's theorem).

Measures on minimally generated compact spaces

Theorem (PBN)

Minimally generated compact spaces carry only separable measures.

Conclusions:

- under *CH* there is a Efimov space admitting only separable measures;
- there is a compactification of ω admitting only separable measures, and without a convergent subsequence of ω;
- there is a compact but not sequentially compact space admitting only separable measures (Haydon's theorem).

More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K. For $A \subseteq K$ let

$$convA = conv\{\delta_a \colon a \in A\},$$

i.e. $convA$ is the set of all probability measures supported by a finite subset of A. Moreover, put

$$S(A) = \bigcup_{\alpha<\omega_1} scl_\alpha(A),$$

where $scl_0(A) = convA$ and $scl_{\alpha+1}(A)$ is the sequential closure of $scl_\alpha(A)$ in the weak* topology of $P(K)$.

More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K. For $A \subseteq K$ let

$$conv A = conv\{\delta_a \colon a \in A\},$$

i.e. $conv A$ is the set of all probability measures supported by a finite subset of A. Moreover, put

$$S(A) = \bigcup_{\alpha<\omega_1} scl_\alpha(A),$$

where $scl_0(A) = conv A$ and $scl_{\alpha+1}(A)$ is the sequential closure of $scl_\alpha(A)$ in the weak* topology of $P(K)$.

More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K. For $A \subseteq K$ let

$$convA = conv\{\delta_a \colon a \in A\},$$

i.e. $convA$ is the set of all probability measures supported by a finite subset of A. Moreover, put

$$S(A) = \bigcup_{\alpha<\omega_1} scl_\alpha(A),$$

where $scl_0(A) = convA$ and $scl_{\alpha+1}(A)$ is the sequential closure of $scl_\alpha(A)$ in the weak* topology of $P(K)$.

More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K. For $A \subseteq K$ let

$$conv A = conv\{\delta_a \colon a \in A\},$$

i.e. $conv A$ is the set of all probability measures supported by a finite subset of A. Moreover, put

$$S(A) = \bigcup_{\alpha<\omega_1} scl_\alpha(A),$$

where $scl_0(A) = conv A$ and $scl_{\alpha+1}(A)$ is the sequential closure of $scl_\alpha(A)$ in the weak* topology of $P(K)$.

More on measures on minimally generated spaces

Sequential closure of the family of finitely supported measures

Denote by $P(K)$ the family of probability measures on K. For $A \subseteq K$ let

$$conv A = conv\{\delta_a \colon a \in A\},$$

i.e. $conv A$ is the set of all probability measures supported by a finite subset of A. Moreover, put

$$S(A) = \bigcup_{\alpha<\omega_1} scl_\alpha(A),$$

where $scl_0(A) = conv A$ and $scl_{\alpha+1}(A)$ is the sequential closure of $scl_\alpha(A)$ in the weak* topology of $P(K)$.

Mazur property

Theorem (Plebanek)

Let K be a compactification of ω, and suppose that

- $P(K) = S(K)$;
- *for every $x \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$, then $\delta_t \in S(Y)$.*

Then $C(K)$ has the Mazur property.

Theorem (Plebanek, PBN)

If K is minimally generated, then $P(K) = S(K)$.

Mazur property

Theorem (Plebanek)

Let K be a compactification of ω, and suppose that

- $P(K) = S(K)$;
- *for every $x \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$, then $\delta_t \in S(Y)$.*

Then $C(K)$ has the Mazur property.

Theorem (Plebanek, PBN)

If K is minimally generated, then $P(K) = S(K)$.

Mazur property

Theorem (Plebanek)

Let K be a compactification of ω, and suppose that

- $P(K) = S(K)$;
- *for every* $x \in K \setminus \omega$ *and every* $Y \subseteq \omega$*, if* $t \in \overline{Y}$*, then* $\delta_t \in S(Y)$.

Then $C(K)$ *has the Mazur property.*

Theorem (Plebanek, PBN)

If K is minimally generated, then $P(K) = S(K)$.

Mazur property

Theorem (Plebanek)

Let K be a compactification of ω, and suppose that

- $P(K) = S(K)$;
- *for every $x \in K \setminus \omega$ and every $Y \subseteq \omega$, if $t \in \overline{Y}$, then $\delta_t \in S(Y)$.*

Then $C(K)$ has the Mazur property.

Theorem (Plebanek, PBN)

If K is minimally generated, then $P(K) = S(K)$.

Gelfand–Phillips property

Theorem (Schlumprecht)

If $\mathfrak{A}/Fin$ is dense in $P(\omega)/Fin$, then $C(Stone(\mathfrak{A}))$ does not have Gelfand-Phillips property.

Theorem (Balcar, Simon, Pelant)

There is a minimally generated Boolean algebra $\mathfrak{A}$ with the above property.

Gelfand–Phillips property

Theorem (Schlumprecht)

If $\mathfrak{A}/Fin$ is dense in $P(\omega)/Fin$, then $C(Stone(\mathfrak{A}))$ does not have Gelfand-Phillips property.

Theorem (Balcar, Simon, Pelant)

There is a minimally generated Boolean algebra $\mathfrak{A}$ with the above property.

Mazur vs. Gelfand-Phillips property

Theorem (Plebanek, PBN)

If $\mathfrak{k} > \mathfrak{h}$, then there is a Banach space satisfying Mazur property but without Gelfand-Phillips property.

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f : \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f\colon \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f : \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f\colon \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f \colon \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| \colon \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Combinatorics

- $\mathfrak{p}$ - pseudo–intersection number;
- $\mathfrak{s}$ - splitting number;
- $\mathfrak{h}$ - distributive number;
- $\mathfrak{k}$ - condensation number.

Condensation

We say that $\mathcal{A} \subseteq P(\omega)$ is condensed if there is a bijection $f\colon \omega \to \omega$ such that $f^{-1}[A]$ has density 1 for every $A \in \mathcal{A}$. Let

$$\mathfrak{k} = \min\{|\mathcal{A}| : \mathcal{A} \text{ is not condensed}\}.$$

Problem

Is it consistent to assume that $\mathfrak{k} > \mathfrak{h}$?

Another example of consistency result

Theorem (PBN)

If $\mathfrak{p} = \mathfrak{c}$, then there is a space without a copy of $\beta\omega$ such that every convergent sequence consists of infinitely many points of character $\mathfrak{c}$.