Measures on small Boolean algebras

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Boolean algebras $\sim$ zerodimensional compacta

A Boolean algebra is **small** if it does not contain any ω_1 independent sequence. Analogously a compact space is small if it cannot be continuously mapped onto $[0,1]^{\omega_1}$.

finitely additive measures $\sim$ Radon measures

A measure μ on a Boolean algebra $\mathbf{B}$ is **small** (of type ω) if there is a countable family $\mathcal{A} \subseteq \mathbf{B}$ such that for every $B \in \mathbf{B}$ we have

$$inf\{\mu(A \triangle B) \colon A \in \mathcal{A}\} = 0$$

Remark: Every big Boolean algebra carries a big measure.

Main question: Is there any small Boolean algebra admitting a big measure?

Fremlin: (1996) $MA + \neg CH \vdash$ small Boolean algebras carry only small measures.

It is not true in general. Under CH there are plenty counterexamples, eg.:

- Haydon (1978),
- Kunen (1981),
- Kunen, Džamonja (1994).

What about particular well-known subclasses of small Boolean algebras?

Boolean algebras	**compact spaces**
interval	ordered
tree	
superatomic	scattered
	monotonically normal
retractive	co-retractive
	Corson compacta
	perfectly normal
	hereditary separable

Koppelberg:

A Boolean algebra is **minimally generated** if there is a maximal chain in the lattice of its subalgebras and this chain happens to be well-ordered.

$$\left.\begin{array}{c}\text{interval}\\ \text{tree}\\ \text{superatomic}\end{array}\right\} \Longrightarrow \text{ minimally generated}$$

Plebanek: Every zerodimensional monotonically normal space is minimally generated.

PBN: Minimally generated Boolean algebras carry only small measures.

Koppelberg: $CH \vdash$ there is a retractive Boolean algebra which is not minimally generated.

PBN: $CH \vdash$ there is a retractive Boolean algebra supporting a big measure.

Question: Can we construct a retractive Boolean algebra with measure of type ω_2?

Question: (Fremlin) Can we construct a perfectly normal compact space with measure of type ω_2?

Question: Is it consistent to assume that every retractive algebra is minimally generated?

Koppelberg: Every minimally generated Boolean algebra has a dense tree subalgebra (equivalently, has the same completion as some interval algebra).

Balcar, Simon, Pelant: $P(\omega)/Fin$ has a dense tree subalgebra.

PBN: $CH \vdash$ retractive algebras do not need to have a dense tree subalgebra.

Question: Is it consistent that every small Boolean algebra has a dense tree algebra?

Motivations are topological. For example

Efimov Problem: Does every compact space without nontrivial convergent sequences contains a copy of $\beta\omega$?

http://www.math.uni.wroc.pl/~pborod/