Startegies and tactics in measure games

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Game For a family of sets $\mathcal{J}$ we consider game $BM(\mathcal{J})$ with two players (**Empty**, who starts the game with an element of $\mathcal{J}$ and **Non-empty**). Players have to choose sets from $\mathcal{J}$ included in the last move of the adversary. Empty wins if the intersection of the game is empty.

Strategy A function $\sigma\colon \mathcal{J}^{<\omega} \longrightarrow \mathcal{J}$ is called *winning strategy* for Nonempty in $BM(\mathcal{J})$ if

$$\bigcap_{n\in\omega} K_n \neq \emptyset,$$

whenever $(K_n)_{n\in\omega}$ is a sequence in $\mathcal{J}$ such that

$$K_{n+1} \subseteq \sigma(K_0, ..., K_n)$$

for every n.

Topology

If (X, τ) is a topological space, we can consider a game $BM(\tau \setminus \{\emptyset\})$. It is usually called a *Choquet* game. It is convenient to say that a topological space X is Choquet if Nonempty has a winning strategy in $BM(X)$.

Theorem (Oxtoby) A nonempty topological space X is a Baire space iff Empty has no winning strategy in the game $BM(X)$

Corollary Every Choquet space is Baire.

Measures

For a measure μ on Σ we can consider a game $BM(\Sigma^+)$. We will say that a measure $\mu|\Sigma$ is *weakly α-favourable* if Nonempty has a winning strategy in $BM(\Sigma^+)$.

Theorem (Fremlin) Every weakly α-favourable measure is perfect.

Explanation A measure (X, Σ, μ) is *perfect* if for every measurable function $f: X \to [0,1]$ and every $E \in Borel([0,1])$ such that $f^{-1}(E) \in \Sigma$ we can find a borel set $B \subseteq E$ such that

$$\mu f^{-1}(E) = \mu f^{-1}(B).$$

Strategy A function $\sigma\colon \mathcal{J}^{<\omega} \longrightarrow \mathcal{J}$ is called *winning strategy* for Nonempty in $BM(\mathcal{J})$ if

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for every n.

Tactic A function $\tau\colon \mathcal{J} \longrightarrow \mathcal{J}$ is called *winning tactic* for Nonempty in $BM(\mathcal{J})$ if

$$\bigcap_{n\in\omega} K_n \neq \emptyset,$$

whenever $(K_n)_{n\in\omega}$ is a sequence in $\mathcal{J}$ such that

$$K_{n+1} \subseteq \tau(K_n)$$

for every n.

Theorem (Debs) There is a class $\mathcal{J}$ for which Nonempty has a winning strategy in $BM(\mathcal{J})$ but doesn't have any winning tactic.

Example Let $Baire$ be the algebra of subsets of $[0,1]$ with the Baire property, and $\mathcal{M}$ the ideal of meager subsets of $[0,1]$. Denote by $\mathcal{J}$ the family $Baire \setminus \mathcal{M}$.

Nonempty has a winning strategy in $BM(\mathcal{J})$.

Nonempty doesn't have a winning tactic in $BM(\mathcal{J})$.

Fact Nonempty doesn't have a winning tactic in $BM(\mathcal{J})$.

Let $\mathcal{U}$ be a countable base for the topology of $[0,1]$, not containing the empty set. Assume for the contradiction that there is a winning tactic for Nonempty in $BM(\mathcal{J})$. Denote it by τ.

For every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ (*good* for U) such that for every $M \in \mathcal{M}$

$$\forall M \in \mathcal{M} \quad \exists N \in \mathcal{M} \quad N \supseteq M \quad V \subseteq^* \tau(U \setminus N).$$

Construct a sequence $(V_n)_{n\in\omega}$ such that V_{n+1} is good for V_n for every n and $\bigcap_{n\in\omega} V_n$ contains at most one point.

The sequence $(V_n)_{n\in\omega}$ is a framework of the play for which Nonempty's tactic fails.

We will say that a measure $\mu|\Sigma$ is *weakly α-favourable* if Nonempty has a winning strategy in $BM(\Sigma^+)$.

We will say that a measure $\mu|\Sigma$ is *α-favourable* if Nonempty has a winning tactic in $BM(\Sigma^+)$.

Problem Is every weakly α-favourable measure α-favourable?

countably compact $\Longrightarrow$ α-favourable $\Longrightarrow$ weakly α-favourable $\Longrightarrow$ perfect

Definition Family of sets $\mathcal{K}$ is *countably compact* if for every sequence $(K_n)_{n\in\omega}$ of sets from $\mathcal{K}$ such that its every finite intersection is non-empty, we have

$$\bigcap_{n\in\omega} K_n \neq \emptyset.$$

Definition (Marczewski) Measure $\mu|\Sigma$ is countably compact if it is inner regular with respect to some countably compact class $\mathcal{K}$, it means for every $E \in \Sigma$ we have

$$\mu(E) = sup\{\mu(K) \colon K \in \mathcal{K}, K \subseteq E\}.$$

Question Does every measure defined on sub-σ-algebra Σ of $Borel([0,1])$ need to be countably compact?

Theorem (Fremlin) Every measure defined on $\Sigma \subseteq Borel([0,1])$ is weakly α-favourable.

Theorem (Plebanek, PBN) A measure $\mu|\Sigma$ (where Σ as above) is countably compact provided there is a family $\{B_\alpha\}_{\alpha<\omega_1}$ of analytic sets, such that μ is regular with respect to the family of those $E \in \Sigma$ for which there is $\alpha < \omega_1$ such that $E \subseteq B_\alpha$ is closed in B_α.

Theorem (Plebanek, PBN) Every measure defined on sub-σ-algebra of $Borel([0,1])$ is an image of a monocompact measure.

Theorem (Fremlin) Every measure defined on $\Sigma \subseteq Borel([0,1])$ is weakly α-favourable.

- for $n \in \omega$ and $\psi \in \omega^n$ denote

  $V(\psi) = \{x \in \mathcal{N} : \ x(k) \leq \psi(k)$ for all $k < n\}$;

- let A_n be n-th move of Nonempty and B_n - n-th move of his adversary;

- let $F_n \in Closed([0,1] \times \omega^\omega)$ such that $\pi(F_n) = A_n$;

- Nonempty will construct inductively collection of functions $(\phi_n)_{n\in\omega}$ from ω^ω such that $\mu^*(Y_n) > 0$, where

$$Y_n = \bigcap_{k=0}^{n} \pi(F_k \cap ([0,1] \times V(\phi_k|n))$$

- Nonempty will construct inductively collection of functions $(\phi_n)_{n\in\omega}$ from ω^ω such that $\mu^*(Y_n) > 0$, where
$$Y_n = \bigcap_{k=0}^{n} \pi(F_k \cap ([0,1] \times V(\phi_k|n))$$
and play the measurable hull of Y_n as B_n;

- consider any sequence $x_n \in Y_n$, which is convergent (to some $x \in [0,1]$);

- fix $k \in \omega$;

- for every $n \geq k$ we can find
$$y_n \in F_k \cap ([0,1] \times V(\phi_k|n))$$
moreover we can assume that $(y_n)_n$ converges to some $y \in \omega^\omega$;

- then $(x, y) \in F_k$ and thus $x \in A_k$ but k was arbitrary.

References:

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