Compact spaces and convergent sequences

Piotr Borodulin-Nadzieja

Instytut Matematyczny
Uniwersytetu Wrocławskiego

Definition A Boolean algebra $\mathfrak{B}$ extends an algebra $\mathfrak{A}$ **minimally** if $\mathfrak{A} \subseteq \mathfrak{B}$ and there is no algebra $\mathfrak{C}$ such that $\mathfrak{A} \subsetneq \mathfrak{C} \subsetneq \mathfrak{B}$.

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Definition A Boolean algebra $\mathfrak{A}$ is **minimally generated** if there is a sequence $(\mathfrak{A}_\alpha)_{\alpha<\kappa}$ such that

- $\mathfrak{A}_0 = \{0, 1\}$;
- $\mathfrak{A}_{\alpha+1}$ extends $\mathfrak{A}_\alpha$ minimally for $\alpha < \kappa$;
- $\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ for limit γ;
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A topological space K is minimally generated if it is a Stone space of a minimally generated Boolean algebra.

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More precisely: If μ is a measure on a minimally generated compact space, then it is a countable sum of weakly uniformly regular measures.

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- Consider a minimally generated Boolean algebra

$$Fin \subseteq \mathfrak{A} \subseteq P(\omega),$$

  such that $\mathfrak{A}$ cannot be extended minimally by a subset of ω.

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Theorem (Haydon) There is a compact but not sequentially compact space carrying only separable measures.

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If MA is assumed, then there is a (minimally generated) compact space without a copy of $\beta\omega$ such that convergent sequences consist only of point of character $\mathfrak{c}$;

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Is there a minimally generated compactification X of ω such that:

- X does not contain a nontrivial convergent sequence of natural numbers;
- for every $x \in X$ there is a sequence $(n_k)_{k\in\omega}$ of natural numbers such that

$$\delta_x = \lim_{k\to\infty} \frac{1}{k}(\delta_{n_1} + ... + \delta_{n_k})?$$

Definition A Banach space E has the Mazur property if every $x^{**} \in E^{**}$ which is weak sequentially continuous on E^* belongs to E.

Definition A Banach space E is said to have the Gelfand–Phillips property if every limited subset of E is relatively norm compact.

A subset A of E is limited if

$$\lim_{n\to\infty} \sup_{x\in A} x_n^*(x) = 0,$$

for every weak* null sequence x_n^* in E^*.

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- there is a Banach space E with the Gelfand–Philips property and without the Mazur property;

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- $C(X)$ is an example of Banach space with the Mazur property and without the Gelfand-Phillips property (if only such X exists).

- for $A \subseteq \omega$ define the asymptotic density function by

$$d(A) = \lim_{n\to\infty} \frac{|A \cap n|}{n},$$

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- we say that M is a condeser of a filter $\mathcal{F}$ on ω if $d_M(F) = 1$ for every $F \in \mathcal{F}$.

- M is a pseudo–intersection of a filter $\mathcal{F}$ on ω if $M \subseteq^* F$ for every $F \in \mathcal{F}$.

$$\mathfrak{p} = min\{|\mathcal{A}| \colon \mathcal{A} \subseteq P(\omega) \text{ generates a filter without a pseudo–intersection }\}.$$

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- it is easy to see that if M is a pseudo–intersection of a filter $\mathcal{F}$, then it is a condenser of this filter. Therefore

$$\mathfrak{p} \leq \mathfrak{k}.$$

If $\mathfrak{A}$ is a minimally generated Boolean algebra such that no ultrafilter on $\mathfrak{A}$ has an infinite pseudo–intersection but every ultrafilter has a condenser, then $X = Stone(\mathfrak{A})$ satisfies the demanded properties.

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$$Con(\mathfrak{p} < \mathfrak{k})?$$

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Theorem If $\mathfrak{h} < \mathfrak{b}$, then there is a minimally generated Boolean algebra such that no ultrafilter on $\mathfrak{A}$ has an infinite pseudo–intersection but every ultrafilter on $\mathfrak{A}$ is feeble.