On properties of measures on unit interval

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Definition 1 Family of sets $\mathcal{K}$ is **countably compact** if for every sequence $(K_n)_{n\in\omega}$ of sets from $\mathcal{K}$ such that its every finite intersection is nonempty, we have

$$\bigcap_{n\in\omega} K_n \neq \emptyset.$$

Definition 2 [*Marczewski*] Measure $\mu|\Sigma$ is countably compact if it is inner regular with respect to some countably compact class $\mathcal{K}$, it means for every $E \in \Sigma$ we have

$$\mu(E) = sup\{\mu(K)\colon K \in \mathcal{K}, K \subseteq E\}.$$

Definition 3 Family of sets $\mathcal{K}$ is **monocompact** if for every **decreasing** sequence $(K_n)_{n\in\omega}$ of sets from $\mathcal{K}$ such that its every finite intersection is nonempty, we have

$$\bigcap_{n\in\omega} K_n \neq \emptyset.$$

Question Does every measure defined on sub-σ-algebra Σ of $Borel([0,1])$ need to be countably compact?

Theorem [*Fremlin*] If Σ is ω_1-generated the answer for the above question is **yes**.

Corollary (CH) Every measure defined on sub-σ-algebra of $Borel([0,1])$ is countably compact.

Theorem [*G. Plebanek, P.B.N.*] Measure $\mu|\Sigma$ (where Σ as above) is countably compact provided there is a family $\{B_\alpha\}_{\alpha<\omega_1}$ of analytic sets, such that μ is regular with respect to the family of those $E \in \Sigma$ for which there is $\alpha < \omega_1$ such that $E \subseteq B_\alpha$ is closed in B_α.

Theorem [*G. Plebanek, P.B.N.*] Every measure defined on sub-σ-algebra of $Borel([0,1])$ is an image of a monocompact measure.

Game For a family of sets $\mathcal{J}$ we consider game $BM(\mathcal{J})$ with two players (**Empty**, who starts the game with an element of $\mathcal{J}$ and **Nonempty**). Players have to choose sets from $\mathcal{J}$ included in the last move of the adversary. Empty wins if the intersection of game is empty.

Definition 4 We will say that $\mu|\Sigma$ is *weakly* α*-favourable* if Σ^+ is weakly α-favourable class, it means Nonempty has a winning **strategy** in $BM(\Sigma^+)$.

Definition 5 We will say that $\mu|\Sigma$ is α*-favourable* if Σ^+ is α-favourable class, it means Nonempty has a winning **tactic** in $BM(\Sigma^+)$.

Theorem [*Fremlin*] Every measure defined on $\Sigma \subseteq Borel([0,1])$ is weakly α-favourable.

Proof:

- for $n \in \omega$ and $\psi \in \omega^n$ denote

  $V(\psi) = \{x \in \mathcal{N} : \ x(k) \leq \psi(k)$ for all $k < n\}$;

- let A_n be n-th move of Nonempty and B_n - n-th move of his adversary;

- let $F_n \in Closed([0,1] \times \omega^\omega)$ such that $\pi(F_n) = A_n$;

- Nonempty will construct inductively collection of functions $(\phi_n)_{n \in \omega}$ from ω^ω such that $\mu^*(Y_n) > 0$, where

$$Y_n = \bigcap_{k=0}^{n} \pi(F_k \cap ([0,1] \times V(\phi_k|n))$$

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and play the measurable hull of Y_n as B_n;

- consider any sequence $x_n \in Y_n$, which is convergent (to some $x \in [0,1]$);

- fix $k \in \omega$;

- for every $n \geq k$ we can find

$$y_n \in F_k \cap ([0,1] \times V(\phi_k|n))$$

moreover we can assume that $(y_n)_n$ converges to some $y \in \omega^\omega$;

- then $(x, y) \in F_k$ and thus $x \in A_k$ but k was arbitrary.

Hierarchy The following implications occurs within the family of measures considered above:

$$\begin{gathered}\text{countably compact}\\ \Downarrow\ \Uparrow\\ \text{monocompact}\\ \Downarrow\\ \alpha\text{–favourable}\\ \Downarrow\\ \text{weakly } \alpha\text{–favourable}\\ \Downarrow\ \Uparrow\\ \text{perfect.}\end{gathered}$$