Minimally generated Boolean algebras

Piotr Borodulin-Nadzieja

Instytut Matematyczny
Uniwersytetu Wrocławskiego

Definition A Boolean algebra $\mathfrak{B}$ extends an algebra $\mathfrak{A}$ **minimally** if $\mathfrak{A} \subseteq \mathfrak{B}$ and there is no algebra $\mathfrak{C}$ such that $\mathfrak{A} \subsetneq \mathfrak{C} \subsetneq \mathfrak{B}$.

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Definition A Boolean algebra $\mathfrak{A}$ is **minimally generated** if there is a sequence $(\mathfrak{A}_\alpha)_{\alpha<\kappa}$ such that

- $\mathfrak{A}_0 = \{0, 1\}$;
- $\mathfrak{A}_{\alpha+1}$ extends $\mathfrak{A}_\alpha$ minimally for $\alpha < \kappa$;
- $\mathfrak{A}_\gamma = \bigcup_{\alpha<\gamma} \mathfrak{A}_\alpha$ for limit γ;
- $\mathfrak{A} = \bigcup_{\alpha<\kappa} \mathfrak{A}_\alpha$.

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A topological space K is minimally generated if it is a Stone space of a minimally generated Boolean algebra.

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Theorem (Koppelberg) There is a small Boolean algebra which is not minimally generated.

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Fact Every big Boolean algebra carries a non-separable measure.

Theorem (Fremlin) $MA(\omega_1)$ implies that small Boolean algebras admit only separable measures.

Theorem Minimally generated Boolean algebras admit only separable measures.

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More precisely: If μ is a measure on a minimally generated Boolean algebra then it is a countable sum of weakly uniformly regular measures.

Corollary:

The following classes of Boolean algebras (spaces) admit only separable measures:

- interval algebras (ordered spaces);
- tree algebras;
- superatomic algebras (scattered spaces);
- monotonically normal spaces.

Corollary:

The following classes of Boolean algebras (spaces) are not included (at least, not in ZFC) in the class of minimally generated Boolean algebras:

- retractive algebras;
- Corson compacta.

Another corollaries:

A length of minimally generated Boolean algebra has to be a limit ordinal but not necessarily a cardinal.

In the realm of minimally generated spaces

$$\text{ccc} = \text{separability}$$

Moreover, if we assume Suslin Conjecture then

$$\text{ccc} = \text{existence of strictly positive measure}$$

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Theorem (Fedorchuk) CH implies the existence of such space.

It is not known if it is consistent to assume that there is no Efimov space.

Problem How to characterize minimally generated spaces?

Theorem (Koppelberg) Every minimally generated space has a tree π-base.

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Question Is it true that every minimally generated space is discretely generated?