ALGEBRA 1R, Problem List 10

Let R and S be commutative rings with 1 and $n \in \mathbb{N}$.

(1) Assume that R is a *Boolean ring* that is for each $r \in R$ we have $r^2 = r$.
  (a) Show that for each $r \in R$ we have $r + r = 0$.
  (b) For each set X, find a natural structure of a Boolean ring on the set of all subsets of X.

(2) Find a monomorphism of rings
$$R \to \mathrm{End}(R, +)$$
and a monomorphism of groups
$$R^* \to \mathrm{Aut}(R, +).$$

(3) Check whether the monomorphism
$$R^* \to \mathrm{Aut}(R, +)$$
from Problem (2) above is an isomorphism in the following cases:
  (a) $R = \mathbb{Q}$,
  (b) $R = \mathbb{R}$,
  (c) $R = \mathbb{C}$.

(4) Show that $R[\![X]\!]$ with the operations given during the lecture is a commutative ring with 1 and that $R[X]$ is a subring of $R[\![X]\!]$.

(5) Show that if R is a domain, then $R[\![X]\!]$ is a domain as well.

(6) Let $F = \sum a_i X^i \in R[\![X]\!]$. Show that $F \in R[\![X]\!]^*$ if and only if $a_0 \in R^*$.

(7) Let R be a domain and
$$P = a_0 + a_1 X + \ldots + a_n X^n \in R[X].$$
Show that $P \in R[X]^*$ if and only if $a_1 = \ldots = a_n = 0$ and $a_0 \in R^*$.

(8) Find R and $a \in R \setminus \{0\}$ such that $1 + aX \in R[X]^*$.

(9) Let $f : R \to S$ be a homomorphism of rings and assume that S is a domain. Show that: if there is $r \in R$ such that $f(r) \neq 0$, then $f(1_R) = 1_S$.

(10) Show that if R is finite, then R is a field if and only if R is a domain.

(11) Give an example of a field which has 4 elements.