ALGEBRA 1R, Problem List 12

Let R be a commutative ring with 1 and K be a field.

(1) Show that 3 is a reducible element and 5 is an irreducible element of the ring $\mathbb{Z}[\sqrt{-2}]$.

(2) Check whether the following element is irreducible in the ring R.
   (a) $7+\sqrt{-5}$, $2+3\sqrt{-5}$, $5+4\sqrt{-5}$, where $R=\mathbb{Z}[\sqrt{-5}]$;
   (b) $-1+7i$, 5, 23, $1+6i$, where $R=\mathbb{Z}[i]$.

(3) Describe (up to being associated) all the irreducible elements in the ring $K[\![X]\!]$.

(4) Show that the ring $K[X^2, X^3]$ is not UFD.

(5) Show that the ring $\mathbb{Z}[\frac{1}{2}]$ is UFD.

(6) Let R be UFD and $a, b \in R$. Show that the ideal $(a) \cap (b)$ is principal.

(7) Let $p > 2$ be a prime number. Show that the following conditions are equivalent:
   (a) p is a reducible element in the ring $\mathbb{Z}[i]$,
   (b) p is a sum of two squares of integer numbers,
   (c) p is congruent to 1 modulo 4.

(8) Show that among prime numbers there are infinitely many:
   (a) irreducible elements of the ring $\mathbb{Z}[i]$,
   (b) reducible elements of the ring $\mathbb{Z}[i]$.