ALGEBRA 1R, Problem List 1

Special Problem Session 12.10.2022 (Wednesday), 12:15–14:00.

For a set X, $\mathcal{P}(X)$ is the set of all subsets of X, and S_X is the set of all bijections $X \to X$.

(1) Give an example of an operation $*$ on the set $\{0,1\}$ such that
$$0 * (0 * 0) \neq (0 * 0) * 0.$$
How many such operations $*$ are there (on this set $\{0,1\}$)?

(2) Assume that $*$ is an associative operation on a finite set A. Show that there is $a \in A$ such that $a * a = a$.

(3) Let $*$ be an operation on X and $a, b, c \in X$. Show that:
  (a) If b and c are neutral elements of $*$, then $b = c$.
  (b) If the operation $*$ is associative, $*$ has a neutral element e, $a * b = e$, and $c * a = e$, then $b = c$.
  (c) If $(X, *)$ is a group with the neutral element e and $a * b = e$, then $b * a = e$.

(4) Let $f : X \to X$. Show that:
  (a) The function f is onto if and only if there is a function $g : X \to X$ such that $f \circ g = \mathrm{id}_X$.
  (b) The function f is one-to-one if and only if there is a function $h : X \to X$ such that $h \circ f = \mathrm{id}_X$.

(5) Let G be a transformation group on X. Show that $\mathrm{id}_X \in G$.

(6) Show that the operation $+$ on the set $\mathbb{R} \cup \{\infty\}$ (defined during the lecture) is associative and has a neutral element, but $(\mathbb{R} \cup \{\infty\}, +)$ is not a group.

(7) Show that if $|X| > 1$, then (X, L) is not a group, where for $a, b \in X$ we have $a\mathrm{L}b = a$.

(8) Show that if X is non-empty, then:
  (a) $(\mathcal{P}(X), \cup)$ is not a group,
  (b) $(\mathcal{P}(X), \cap)$ is not a group.

(9) Show that the group S_X is commutative if and only if $|X| < 3$.

(10) Check whether the following operation $*$ on the following set A is associative, commutative and whether it has a neutral element. Check also whether $(A, *)$ is a group.
  (a) $A = \mathbb{Q} \setminus \{0\}$; $a * b = \frac{a}{b}$.
  (b) $A = \mathbb{R}$; $x * y = x + y + 2$.
  (c) $A = \mathbb{N}_+$; $m * n = \mathrm{GCD}(m, n)$.
  (d) $A = \mathbb{N}_+$; $m * n = \mathrm{LCM}(m, n)$.
  (e) A is the plane; $P * Q$ is the middle point of the interval with end-points P, Q.
  (f) A is the plane; $P * Q$ is the image of the point P under the reflection across the point Q.