ALGEBRA 1R, Problem List 5

Let G be a group and $n \in \mathbb{N}_{>0}$.

(1) Describe the orbits of the natural action of $\mathrm{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$.

(2) Let $(A, +)$ be a commutative group.

   (a) Show that the following formula:
   $$\forall a \in A \qquad 0 \cdot a = a, \quad 1 \cdot a = -a$$
   gives an action of $\mathbb{Z}_2$ on A by automorphisms.

   (b) Describe the homomorphism
   $$\Psi : \mathbb{Z}_2 \longrightarrow \mathrm{Aut}(A),$$
   which corresponds to the action from Item (a) above.

   (c) For which groups A, the homomorphism Ψ from Item (b) above is a monomorphism?

(3) Asume that there is $g \in G \setminus \{e\}$ such that $\mathrm{ord}(g) \neq 2$. Show that:
$$\mathrm{Aut}(G) \neq \{\mathrm{id}_G\}.$$

(4) Show that:
$$\mathrm{Aut}(\mathbb{Z}_2 \times \mathbb{Z}_2) \cong S_3.$$

(5) Describe the center of S_3 and the center of D_4.

(6) For $n \geqslant 3$, describe the conjugacy class of (123) in S_n.

(7) Describe all conjugacy classes in S_n.

(8) Let $H \leqslant G$. Show that:
$$|G/H| = |H \backslash G|.$$

(9) Show that all automorphims of S_3 are inner.

(10) Show that if $H \leqslant G$ and $[G : H] = 2$, then $H \trianglelefteq G$ (that is: for each $g \in G$, we have $gH = Hg$).

(11) For $n > 1$, show that $T_n(\mathbb{R})$ is not a normal subgroup of $\mathrm{GL}_n(\mathbb{R})$.

(12) Give an example of G and $N \trianglelefteq H \trianglelefteq G$ such that $N \ntrianglelefteq G$.