ALGEBRA 1R, Problem List 7

Let H, G be groups and p, q be prime numbers.

(1) Let $H_1 \trianglelefteq H, G_1 \trianglelefteq G$. Show that $H_1 \times G_1 \trianglelefteq H \times G$ and (using the fundamental theorem on group homomorphisms) that:
$$(H \times G)/(H_1 \times G_1) \cong (H/H_1) \times (G/G_1).$$
(2) Let $\varphi : G \to \mathrm{Aut}(H)$ be an action. Show that the following conditions are equivalent.
 (a) The group $H \rtimes_{\varphi} G$ is commutative.
 (b) The groups H, G are commutative and the action φ is trivial.
(3) Let $\Psi : G \to H$ be an epimorphism. Assume that there is a *section* of Ψ, that is a homomorphism $s : H \to G$ such that $\Psi \circ s = \mathrm{id}_H$. Show that:
$$G \cong \ker(\Psi) \rtimes H.$$
(4) Assume that $|G| = pq$ and $p < q$. Show the following.
 (a) We have:
$$G \cong \mathbb{Z}_q \rtimes \mathbb{Z}_p.$$
 (b) If p does not divide $q - 1$, then we have:
$$G \cong \mathbb{Z}_{pq}.$$
 (c) If p divides $q - 1$, then there is a non-commutative group of order pq.
(5) Classify (up to an isomorphism) all groups of order smaller than 12.
(6) Assume that H is a unique Sylow p-subgroup of G. Show that $H \trianglelefteq G$.
(7) Assume that H is a p-subgroup of G and that H is a normal subgroup. Show that H is contained in each Sylow p-subgroup of G.
(8) Assume that $|G| = 196$. Show that G has a normal subgroup of order 49.
(9) Assume that $|G| = 36$. Show that there is $N \trianglelefteq G$ such that $N \neq \{e\}$ and $N \neq G$.
(10) Find all Sylow p-subgroups of S_p. Conclude the following (Wilson's Theorem):
$$(p - 1)! \equiv -1 (\mathrm{mod}\ p).$$
(11) Show that there is a monomorphism
$$D_4 \longrightarrow S_4.$$
(12) Show that there is no monomorphism
$$Q_8 \longrightarrow S_4.$$
(13) Describe all Sylow p-subgroups of S_4 for all prime numbers p.