Geometria algebraiczna, Problem List 1

Let X be a non-empty topological space, V be an affine variety, $n \in \mathbb{N}_{>0}$, K be an algebraically closed field and $\mathbb{A}^n = K^n$.

(1) Show that if X is Noetherian, then X is *quasi-compact*, i.e. every open covering of X contains a finite open subcovering of X.
(2) Show that if $Y \subseteq X$, then $\dim(Y) \leqslant \dim(X)$, where Y has the subspace topology.
(3) Give an example of X such that X is Noetherian and $\dim(X) = \infty$.
(4) Show that X is Hausdorff and Noetherian if and only if X is finite with the discrete topology.
(5) Show that if X is irreducible and Hausdorff, then $|X| = 1$.
(6) Let $Y \subseteq X$. Show that Y is irreducible (as a topological space with the topology induced from X) if and only if the closure of Y is irreducible.
(7) Show that if X is Noetherian and T_1 (singletons are closed), then

$$\dim(X) = 0 \text{ if and only if } X \text{ is finite.}$$

(8) Suppose that X is irreducible, $Y \subseteq X$ is closed and $\dim(Y) = \dim(X) < \infty$. Show that $Y = X$.
(9) Suppose that $X_1, \ldots, X_n, Y_1, \ldots, Y_m$ are closed irreducible subsets of X such that:

$$X_1 \cup \ldots \cup X_n = Y_1 \cup \ldots \cup Y_m$$

and for all $k \neq l$, we have $X_k \nsubseteq X_l$ and $Y_k \nsubseteq Y_l$. Show that $n = m$ and that there is $\sigma \in S_n$ such that

$$X_1 = Y_{\sigma(1)}, \ldots, X_n = Y_{\sigma(n)}.$$

(10) Suppose that $X = X_1 \cup \ldots \cup X_k$, where $X_1, \ldots, X_k$ are closed substes of X. Show that:

$$\dim(X) = \max\left(\dim(X_1), \ldots, \dim(X_k)\right).$$

(11) Show that the Zariski topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ differs from the product topology (coming from the Zariski topology on $\mathbb{A}^1$).
(12) Let k be an infinite field and $F, G \in k[X_1, \ldots, X_n]$. Show that if F and G define the same polynomial functions from k^n to k, then $F = G$. Find a counterexample for k being the finite field of two elements.
(13) Assume $K = \mathbb{C}$. Find $F \in \mathbb{R}[X, Y]$ which is irreducible (in the ring $\mathbb{R}[X, Y]$) such that $V(F) \cap \mathbb{R}^2$ is non-empty and not irreducible (with the topology induced from $\mathbb{A}^2 = \mathbb{C}^2$).
(14) Assume that X is irreducible and $U \subseteq X$ is open and non-empty. Show that U is dense in X.

(15) Assume that $V_i \subseteq \mathbb{A}^n$ and $A, A_i \subseteq K[X_1, \ldots, X_n]$ for $i \in I$. Show the following:
  (a) $V(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} V(A_i)$;
  (b) $V(A) = V((A))$ ((A) is the ideal in $K[X_1, \ldots, X_n]$ generated by A);
  (c) $I(\bigcup_{i \in I} V_i) = \bigcap_{i \in I} I(V_i)$;
  (d) $A \subseteq I(V(A))$.

(16) Show that each plane curve V is an affine curve (that is: $\dim(V) = 1$).

(17) Let
$$V = \{(t, t^2, t^3) \in \mathbb{A}^3 \mid t \in K\}.$$
Show that V is an affine curve.

(18) Let
$$V = V(X^2 - YZ, XZ - X) \subset \mathbb{A}^3.$$
Describe the decomposition of V into irreducible components.

(19) Show that if $W \subset \mathbb{A}^n$ is finite, then $K[W] = \mathrm{Func}(W, K)$.

(20) Let $V = V(YX - 1) \subset \mathbb{A}^2$. Show that the K-algebras $K[V]$ and $K[\mathbb{A}^1]$ are not isomorphic (even as rings!).

(21) Let $F \in K[X, Y]$ be irreducible and of degree 2. Show that the K-algebra $K[X, Y]/(F)$ is isomorphic with $K[X, 1/X]$ or with $K[X]$.

(22) Let $V = V(Y^2 - X^3)$. Show that $K[V]$ is not UFD.

(23) Let R be PID which is not a field. Show that $\dim(R) = 1$.