Geometria algebraiczna, Problem List 4

Let R be a ring, K be an algebraically closed field, V be an affine algebraic variety over K.

(1) Assume that V is a plane curve. Show that $|V| = |K|$.
(The above result is true for any infinite algebraic set over K. How much can you generalize the above result?)

(2) Assume that $f \in K(V)$. Show that $\mathrm{dom}(f)$ is a Zariski open subset of V.

(3) Assume that R is a UFD and let $f \in R_0$. Show the following.
  (a) There are $f_1, f_2 \in R$ such that $f = f_1/f_2$, where f_1, f_2 have no common irreducible divisors in R.
  (b) For any pairs (f_1, f_2), (g_1, g_2) as in Item (a) above, we have
$$f_1/g_1 \in R^*, \qquad f_2/g_2 \in R^*.$$
  (c) If $R = K(V)$ and f_1, f_2 are as in Item (a) above, then:
$$\mathrm{dom}(f) = V \setminus V(f_2).$$

(4) Show that the following are equivalent.
  (a) R is a local ring.
  (b) $R \setminus R^*$ is closed under addition.
  (c) $R \setminus R^*$ is an ideal of R.
  (d) $R \setminus R^*$ is a unique maximal ideal of R.

(5) Show that if P is a prime ideal of a domain R, then the ring R_P is local and PR_P is a maximal ideal of R_P.

(6) Show that for any $v \in V$, we have:
$$\mathcal{O}_{V,v} = K[V]_{I_V(v)},$$
$$\mathfrak{m}_{V,v} = I_V(v)K[V]_{I_V(v)}.$$

(7) Assume that R is a domain and show that:
$$R = \bigcap\{R_{\mathfrak{m}} \mid \mathfrak{m} \text{ is a maximal ideal of } R\}.$$

(8) Show that:
$$K[V]^* = \{f \in K[V] \mid (\forall v \in V)(f(v) \neq 0)\}.$$

(9) Let $\alpha : R_1 \to R_2$ be a homomorphism of domains and $S_i \subset R_i$ be multiplicative subsets such that $(R_i)^* \subseteq S_i$ ($i \in \{1, 2\}$). Show that α extends to a ring homomorphism
$$(R_1)_{S_1} \to (R_2)_{S_2}$$
if and only if $\alpha(S_1) \subseteq S_2$.