Geometria algebraiczna, Problem List 5

Let K be an algebraically closed field and $m, n, k > 0$.

(1) Let T be a domain. We define:

$$\partial : T[X] \to T[X],\ \partial(a_0 + a_1X + \ldots + a_{n-1}X^{n-1} + a_nX^n) = a_1 + \ldots + (n-1)a_{n-1}X^{n-2} + na_nX^{n-1}.$$

Show that:
 (a) the function ∂ is a derivation;
 (b) if $\mathrm{char}(T) = 0$, then $\partial^{-1}(0) = T$;
 (c) if $\mathrm{char}(T) = p > 0$, then $\partial^{-1}(0) = T[X^p]$.

(2) Suppose the following:
- $G_1, \ldots, G_k, F_1, \ldots, F_m \in K[X_1, \ldots, X_n]$;
- $G_1, \ldots, G_k \in (F_1, \ldots, F_m)$;
- $\bar{F} := (F_1, \ldots, F_m),\ \bar{G} := (G_1, \ldots, G_k),\ v \in V(\bar{F})$.

Show that each row of the matrix $J_{\bar{G}}(v)$ is a K-linear combination of the rows of the matrix $J_{\bar{F}}(v)$.

(3) Let $F_1, \ldots, F_n \in K[X_1, \ldots, X_n]$ and

$$\bar{F} = (F_1, \ldots, F_n) : \mathbb{A}^n \to \mathbb{A}^n$$

be a morphism.
 (a) Show that if $\bar{F}$ is an isomorphism, then $\det(J_{\bar{F}}) \in K^*$.
 (b) What do you think about the converse implication?

(4) Assume that $K = \mathbb{C}$ and $V \subseteq \mathbb{A}^n$ is a smooth algebraic variety. Show that V is a complex submanifold of $\mathbb{C}^n$ (or a differentiable submanifold of $\mathbb{R}^{2n}$). In particular, V becomes a manifold in the sense of differential geometry.

(5) Let P be a prime ideal of a domain R. Show the following.
 (a) We have an R-algebra isomorphism:

$$(R/P)_0 \cong_R R_P/PR_P.$$

 (b) The quotients P/P^2 and $PR_P/(PR_P)^2$ have natural structures of R/P-modules.
 (c) If the ideal P is maximal, then we have an R/P-module isomorphism:

$$P/P^2 \cong_{R/P} PR_P/(PR_P)^2.$$

(6) Assume that $F, G \in K[X, Y]$ are irreducible and F does not divide G. Let $V = V(FG) \subseteq \mathbb{A}^2$ and $a \in V$ be such that $F(a) = G(a) = 0$. Show that a is a singular point of V.

(7) Let $F \in K[X, Y]$ and $V = V(F) \subseteq \mathbb{A}^2$. Show that:
 (a) if $V(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})$ is finite, then $\sqrt{(F)} = (F)$ and $I(V) = (F)$;
 (b) if $V(F, \frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}) = \emptyset$, then V is a smooth algebraic variety.

(8) Suppose that $\mathrm{char}(K) \neq 2$. For $F \in K[X, Y]$ given below, find the singular points of $V(F)$ and show the curve $V(F)$ on the picture below.
 (a) $F = Y^4 + X^4 - X^2$.
 (b) $F = Y^6 + X^6 - XY$.
 (c) $F = Y^4 + X^4 + Y^2 - X^3$.
 (d) $F = Y^4 + X^4 - X^2Y - XY^2$.