Geometria algebraiczna, Problem List 6

Let K be an algebraically closed field, $n \in \mathbb{N}_{>0}$ and $V \subseteq \mathbb{A}^n$ be an algebraic variety.

(1) Show that there is a unique derivation

$$\partial : K[X_1, \ldots, X_n] \to K[X_1, \ldots, X_n, X_1', \ldots, X_n']$$

such that $\partial(K) = \{0\}$ and for $i \in \{1, \ldots, n\}$, we have $\partial(X_i) = X_i'$.

(2) Show that for ∂ from Problem (1) above and $F \in K[X_1, \ldots, X_n]$, we have:

$$\partial(F) = \sum_{i=1}^{n} \frac{\partial F}{\partial X_i} X_i'.$$

(3) Show that if

$$(F_1, \ldots, F_m) = (H_1, \ldots, H_k) \trianglelefteq K[X_1, \ldots, X_n],$$

then for ∂ from Problem (1) above, we have:

$$(F_1, \ldots, F_m, \partial(F_1), \ldots, \partial(F_m)) = (H_1, \ldots, H_k, \partial(H_1), \ldots, \partial(H_k)).$$

(4) Show that if V is smooth, then TV is also smooth.

(5) Let $K = \mathbb{C}$ and assume that V is smooth. We know that V and TV have natural structures of differentiable manifolds. Let $\mathcal{T}V$ denote the tangent bundle in the sense of differential geometry. Show that TV is diffeomorphic to $\mathcal{T}V$ and that this diffeomorphism commutes with the projection maps

$$\pi_V : TV \to V, \qquad \mathcal{T}V \to V.$$

(6) Assume that $0 = (0, \ldots, 0) \in V$. We define the following K-bilinear map:

$$\Psi : K^n \times K[X_1, \ldots, X_n] \to K, \qquad \Psi(x, F) = \partial F(0, x).$$

Show the following:

(a) $\Psi(\pi_V^{-1}(0) \times I(V)) = 0$;

(b) $\Psi(K^n \times I(0)^2) = 0$;

(c) the following K-bilinear map induced (using (a) and (b)) from Ψ

$$\widetilde{\Psi} : \pi_V^{-1}(0) \times I_V(0)/I_V(0)^2 \to K$$

is nondegenerate.

(7) Let R be UFD, $r \in R$ be irreducible and $L = R_0$. We define:

$$v_r : L^* \to \mathbb{Z}, \qquad v_r(\alpha) = n \text{ for } \alpha = r^n \frac{a}{b}, \text{ where } a, b \in R \text{ and } r \nmid ab.$$

For any $\alpha, \beta \in L^*$, show the following:

(a) if $\alpha + \beta \in L^*$, then $v_r(\alpha + \beta) \geqslant \min(v_r(\alpha), v_r(\beta))$;

(b) $v_r(\alpha\beta) = v_r(\alpha) + v_r(\beta)$;

(c) $v_r(L^*) = \mathbb{Z}$.

(8) Let $(R, \mathfrak{m})$ be DVR and v_R the valuation given by a uniformizing parameter for R. Show that for any $a \in R\setminus\{0\}$, we have $v_R(a) = n$, where $a \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$ (we set $\mathfrak{m}^0 := R$).

(9) Let v be a (discrete) valuation on a field L. We define:

$$\mathcal{O}_v := \{x \in L \mid v(x) \geqslant 0\}, \quad \mathfrak{m}_v := \{x \in L \mid v(x) > 0\}.$$

Show that $(\mathcal{O}_v, \mathfrak{m}_v)$ is DVR and $v = v_{\mathcal{O}_v}$.