Berkovich spaces, Problem List 1

Let F be a field and $\|\cdot\| : F \to \mathbb{R}_{\geqslant 0}$ be a function.

1. Suppose that $\|0\| = 0$ and that for all $a, b \in F$ we have
   - $\|ab\| = \|a\|\|b\|$,
   - $\|1 + a\| \leqslant 1 + \|a\|$.

   Show that $\|\cdot\|$ is a semi-norm.

2. Suppose that $\|\cdot\|$ is a semi-norm on F which is equivalent to the trivial norm. Show that $\|\cdot\|$ is the trivial norm.

3. Assume that $\|\cdot\|$ is a norm. Show that the following are equivalent.
   (a) The norm $\|\cdot\|$ is Archimedean.
   (b) For all $x \in F^*$ there is $n \in \mathbb{N}$ such that $\|n \cdot x\| > 1$.
   (c) There is $n \in \mathbb{N}$ such that $\|n \cdot 1\| > 1$.
   (d) The function
   $$\mathbb{N} \ni n \mapsto \|n \cdot 1\| \in \mathbb{R}$$
   is unbounded.

4. Show that if $\|\cdot\|$ is a norm which is equivalent to an Archimedean norm, then $\|\cdot\|$ is Archimedean.

5. Suppose that $\|\cdot\|$ is an Archimedean norm on F. Show that $\mathrm{char}(F) = 0$.

6. Let p, q be different prime numbers. Show that $|\cdot|_p$ and $|\cdot|_q$ are non-equivalent non-Archimedean norms.

7. Suppose that $\|\cdot\|$ is a non-Archimedean non-trivial norm on $\mathbb{Q}$. Show that
   (a) there is a prime number p such that
   $$p\mathbb{Z} = \{a \in \mathbb{Z} \mid \|a\| < 1\};$$
   (b) the norm $\|\cdot\|$ is equivalent to the p-adic norm $|\cdot|_p$.

8. Suppose that $\|\cdot\|$ is an Archimedean norm on $\mathbb{Q}$. Show that $\|\cdot\|$ is equivalent to the absolute value $|\cdot|_\infty$.

9. Suppose that $\|\cdot\|$ is a non-trivial norm on $F(X)$ which is trivial on F. Show that $\|\cdot\|$ is equivalent to $|\cdot|_\infty$ or there is an irreducible polynomial $f \in F[T]$ such that $\|\cdot\|$ is equivalent to $|\cdot|_f$.

10. Let $\|\cdot\|$ be a norm, $(F, d_{\|\cdot\|})$ be the corresponding metric space and $(\widehat{F}, \widehat{d}_{\|\cdot\|})$ be its (metric) completion. Find a structure $\widehat{+}, \widehat{\cdot}, \widehat{\|\cdot\|}$ of a normed field on $\widehat{F}$ such that
    $$\widehat{d}_{\|\cdot\|} = d_{\widehat{\|\cdot\|}}.$$
    Show that the natural map $F \to \widehat{F}$ is a normed field homomorphism.

11. Show that $F((T))$ is complete (with the T-adic norm).