Berkovich spaces, Problem List 2

Let $(k, |\cdot|)$ be an algebraically closed complete normed field and let $\mathbb{A}^1_{\text{Berk}}$ denote the Berkovich affine line over k.

1. Let $\|\cdot\|$ be a norm on a domain R and $a, b \in R$. Show that:

   (a) if $\|a\| > \|b\|$ then $\|a+b\| = \|a\|$,

   (b) the function $\|\cdot\| : R \to \mathbb{R}$ is continuous,

   (c) the norm $\|\cdot\|$ uniquely extends to a norm on the field of fractions of R.

2. For any $x \in \mathbb{A}^1_{\text{Berk}}$, show that x is a point of type (1) if and only if x is not a norm.

3. Show directly (that is without using Ostrowski's theorem) that there are no norms on $\mathbb{C}[T]$ which extend the absolute value on $\mathbb{C}$.

4. Show that the topology on $\mathbb{A}^1_{\text{Berk}}$ coincides with the topology induced from $\mathbb{R}^{k[T]}$ (considered with the Tychonoff product topology).

5. Assume that the norm $|\cdot|$ is non-Archimedean. Show that:

   (a) any point of a ball (closed or open in the metric space k) is its center,

   (b) each open ball is a clopen set,

   (c) each closed ball of a non-zero radius is a clopen set.

6. Assume that $|\cdot|$ is the trivial norm. Show that the topological space $\mathbb{A}^1_{\text{Berk}}$ is uniquely path-connected, contractible and locally compact.

7. Let ξ be a nested family of closed balls and assume that $B := \bigcap \xi \neq \emptyset$. Show that:

   (a) B is a closed ball,

   (b) $|\cdot|_\xi = |\cdot|_B$.

8. Assume that k is a *separable* topological space (i.e. k has a countable dense subset). Show that k is not spherically complete.