Berkovich spaces, Problem List 3

Let $(k, |\cdot|)$ be an algebraically closed complete non-Archimedean non-trivial normed field and let $\mathbb{A}^1_{\mathrm{Berk}}$ denote the Berkovich affine line over k.

1. Show that
$$\mathcal{O}_{\mathbb{Q}_p} \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$$
(an isomorphism of topological rings).

2. Let D, D' be closed balls in k. Show that $|\cdot|_D \leqslant |\cdot|_{D'}$ if and only if $D \subseteq D'$.

3. Let $|\cdot|_0 \geqslant |\cdot|_1 \geqslant |\cdot|_2 \geqslant \dots$ be semi-norms on a ring R. For $r \in R$ we define
$$|r| := \inf_i |r|_i.$$
Show that $|\cdot|$ is a semi-norm on R.

4. Let ξ be a nested set of balls, $B := \bigcap \xi$ and assume that $B \neq \emptyset$. Show that:

   (a) B is a closed ball,

   (b) $|\cdot|_\xi = |\cdot|_B$.

5. Let $x \in \mathbb{A}^1_{\mathrm{Berk}}$. Show that x is of type (1) if and only if the radius of x is 0.

6. Let $a \in k$ and $r \in \mathbb{R}_{\geqslant 0}$. Show that
$$\zeta_{a,r}(T - a) = r.$$

7. Let $r < r'$ be non-negative real numbers and $a \in k$. Show that $[\zeta_{a,r}, \zeta_{a,r'}]$ (an interval in $\mathbb{A}^1_{\mathrm{Berk}}$) is homeomorphic to $[r, r']$ (an interval in $\mathbb{R}$).